Graphs and Network Flows
ISE 411

Lecture 8

Dr. Ted Ralphs

Why study shortest path problems?

- They arise frequently in practice and as subproblems in higher-level algorithms.
- They are easy to solve efficiently.
- They capture most essential features of the broader class of network flow models.
- SPP is a good starting point for introducing ideas for designing algorithms with good worst-case performance.

Example: The Kevin Bacon Number

- Start with any actor or actress who has been in a movie and connect him or her to Kevin Bacon in the smallest number of links.
- Two people are linked if they have appeared in a movie together.
- Play on the web! `http://OracleOfBacon.org`.
- Example
  - Alfred Hitchcock and Orson Welles: *Show Business At War (1943)*
  - Orson Welles and Jack Nicholson: *A Safe Place (1971)*
  - Jack Nicholson and Kevin Bacon: *A Few Good Men (1992)*
- How is the shortest path problem related to the Oracle of Kevin Bacon?

Formal Definition

Definition 1. *Given a directed network* $G = (N, A)$ *with an arc length* c_{ij} *associated with each arc* $(i, j) \in A$ *and a distinguished node* s*, the* shortest path problem *is to determine a shortest length directed path from node* s *to every node* $i \in N - \{s\}$*.*

- The length of a directed path is the sum of the lengths of arcs in the path.
- $A(i)$ is the arc adjacency list of node i
- $C = \max\{c_{ij} : (i, j) \in A\}$

Variations

- Single-Source: from one node to every other
  - Non-negative arc lengths
  - Arbitrary arc lengths
- All Pairs: from every node to every other
- Maximum Capacity
- Maximum Reliability
- . . .

Assumptions

- The network is directed.
- All arc lengths are integers.
- There is a directed path from node s to every other node in the network.
- There are no directed cycles with negative length.

Shortest Paths Tree

Associated with any shortest path problem is a directed out-tree called the *shortest path tree* rooted at node s with the property that the unique path from node s to any node is a shortest path to that node.

Property 1. *If the path* $s = i_1 - i_2 - \cdots - i_h = k$ *is a shortest path from node* s *to node* k*, then for every* $q = 2, 3, \cdots, h-1$*, the subpath* $s = i_1 - i_2 - \cdots - i_q$ *is a shortest path from the source node to node* i_q*.*

Property 2. *Let the vector* d *represent the shortest path distances. Then a directed path* P *from the source node to node* k *is a shortest path if and only if* $d(j) = d(i) + c_{ij}$ *for every arc* $(i, j) \in P$*.*

From these two properties, we can prove that a shortest paths tree always exists.

Application (Ahuja et al., 4.13)

- The owner of a home for senior citizens has decided to offer shuttle bus service for her residents to downtown Bethlehem.
- She has interviewed a number of potential drivers whose available hours and wages are shown in the table.
- She needs to ensure that at least one driver is on duty for each hour between 9 A.M. and 5 P.M., and she would like to pay as little as possible.
- How should we solve her scheduling problem?

Duty Hours	9 - 1	9 - 11	12 - 3	12 - 5	2 - 5	1 - 4	4 - 5
Cost	30	18	21	38	20	22	9

Shortest Path Algorithms

- Label Setting
  - one label becomes permanent during each iteration
  - acyclic with arbitrary arc lengths OR non-negative arc lengths
- Label Correcting
  - all labels are temporary until last iteration
  - more general graphs including negative arc lengths
- Both are iterative; they differ in label update procedure and convergence procedure.

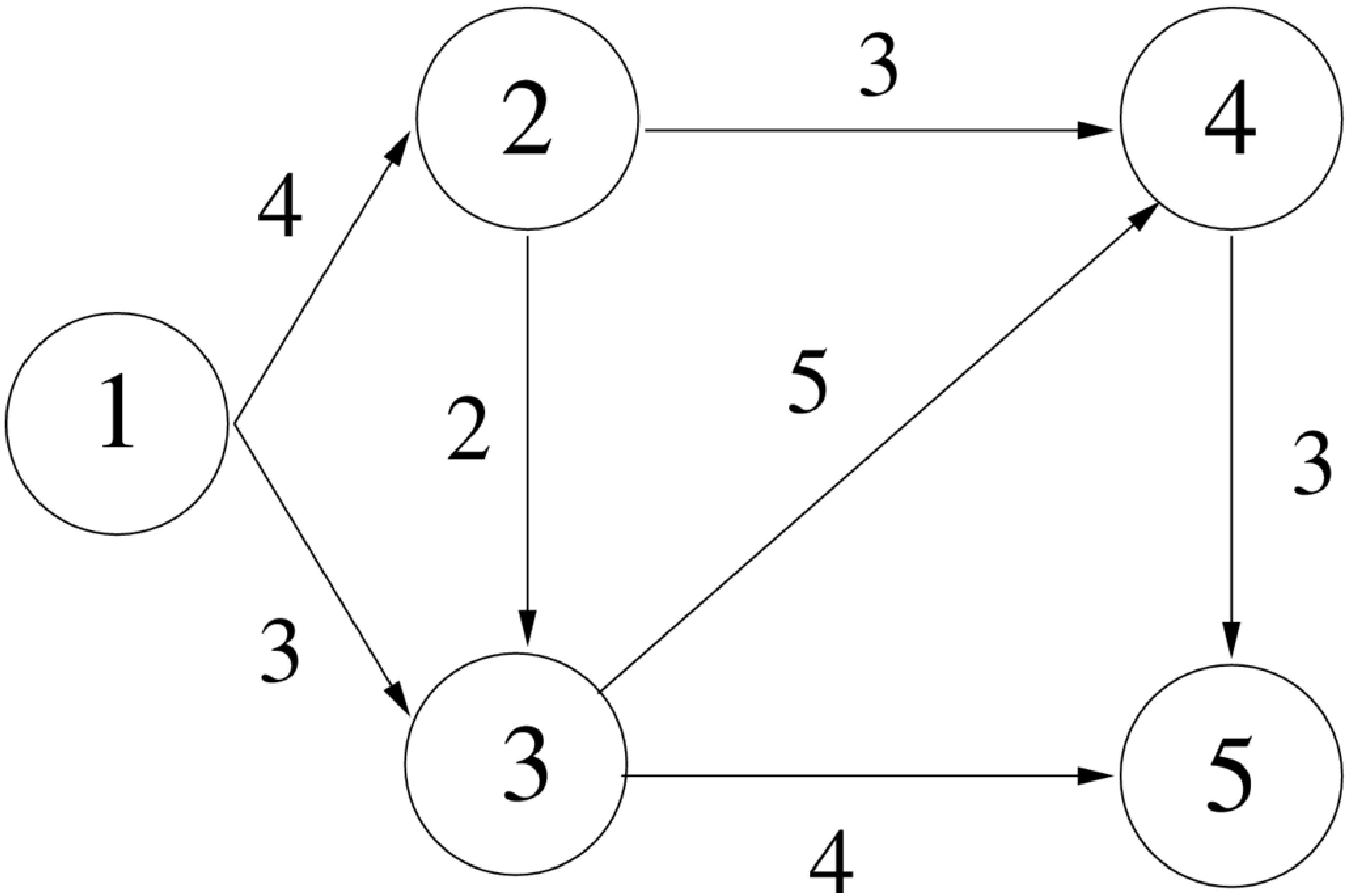
Solving SPP in Acyclic Networks

Reaching Algorithm

Input: An acyclic network $G = (N, A)$ and a vector of arc lengths $c \in \mathbb{Z}^A$
Output: $d(i)$ is the length from of a shortest path from node s to node i

$d(s) \leftarrow 0$
for $i \in N - \{s\}$ **do**
  $d(i) \leftarrow \infty$
end for
Determine a topological order `order[]` of the nodes
for $k = 1$ to n **do**
  $i \leftarrow$ `order`$[k]$
  for $(i, j) \in A(i)$ **do**
    if $d(j) > d(i) + c_{ij}$ **then**
      $d(j) \leftarrow d(i) + c_{ij}$
    end if
  end for
end for

Example of Reaching Algorithm



Proof of Correctness for Reaching Algorithm

Claim: When the algorithm examines a node, its distance label is optimal.

Proof

Base Case: When we examine node 1, $d(1) = 0$ is correct. When we examine node 2, $d(2) = d(1) + c_{12}$ is correct because only node 1 can be inbound to node 2 (topological ordering).

Induction: Suppose that the algorithm has examined nodes $1, 2, \cdots, k$ and the distance labels are correct. Show that the distance label for node $k + 1$ is correct.

Let the shortest path from s to $k + 1$ be $s - i_1 - i_2 - \cdots - i_h - i_k + 1$.