Graphs and Network Flows
ISE 411

Lecture 7

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References for Today's Lecture

- Required reading
  - Chapter 20
- References
  - AMO Chapter 13
  - CLRS Chapter 23

Minimum Spanning Trees

- Optimality Conditions
- Kruskal's Algorithm
- Prim's Algorithm

Combinatorial Optimization

- A *combinatorial optimization problem* consists of
  - a ground set of elements E,
  - an associated set $\mathcal{F}$ of subsets of E called the *feasible subsets*.
  - A cost vector $\mathbb{R}^E$.
- The cost $c(S)$ of a feasible subset is $\sum_{s \in S} c_s$.
- The goal is to find a subset of minimum cost.

Minimum Spanning Trees

- Recall that *a spanning tree T* of G is a connected acyclic subgraph that spans all the nodes of G.
- The total cost of a spanning tree is the sum of the costs of the arcs in the tree.
- Given an undirected graph $G = (N, A)$ with n nodes and m arcs and with a length or cost c_{ij} associated with each arc $(i, j) \in A$, the minimum spanning tree problem is to find a spanning tree with the smallest total cost (length).
- This is a combinatorial optimization problem.

Optimality Conditions

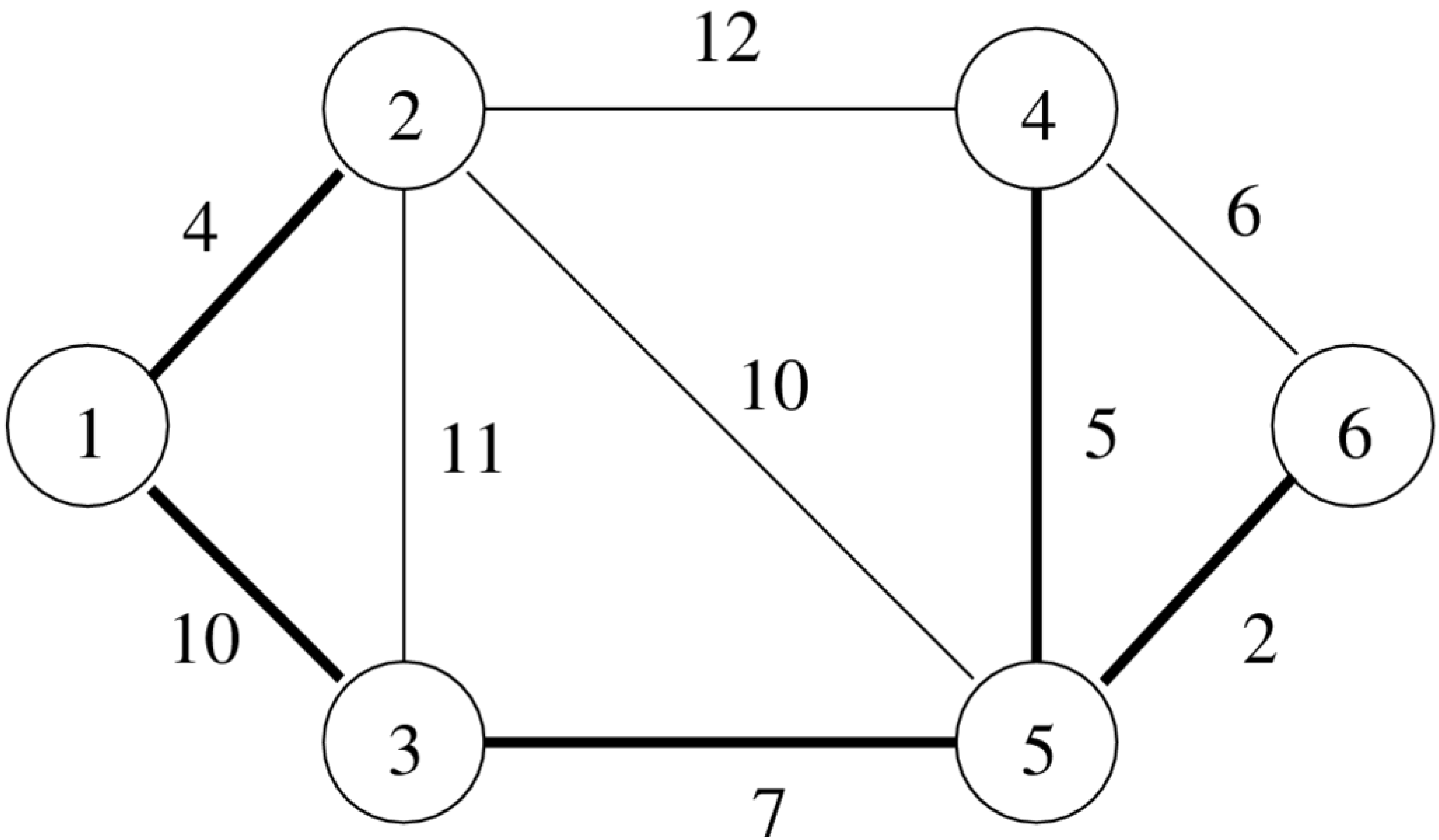
- Cut Optimality Conditions
- Path Optimality Conditions

Properties of a Spanning Tree

- For every non-tree arc (k, l), a spanning tree T contains a unique path from node k to node l. The arc (k, l) together with the unique path defines a cycle.
- If we delete any tree arc (i, j) from a spanning tree, we partition the node set into two subsets, which define a cut in the graph.

Cut Optimality Conditions

Theorem 1. [13.1] *A spanning tree T^* is a minimum spanning tree if and only if for every tree arc $(i,j) \in T^*$, $c_{ij} \leq c_{kl}$ for every arc (k,l) contained in the cut formed by deleting arc (i,j) from T^*.*



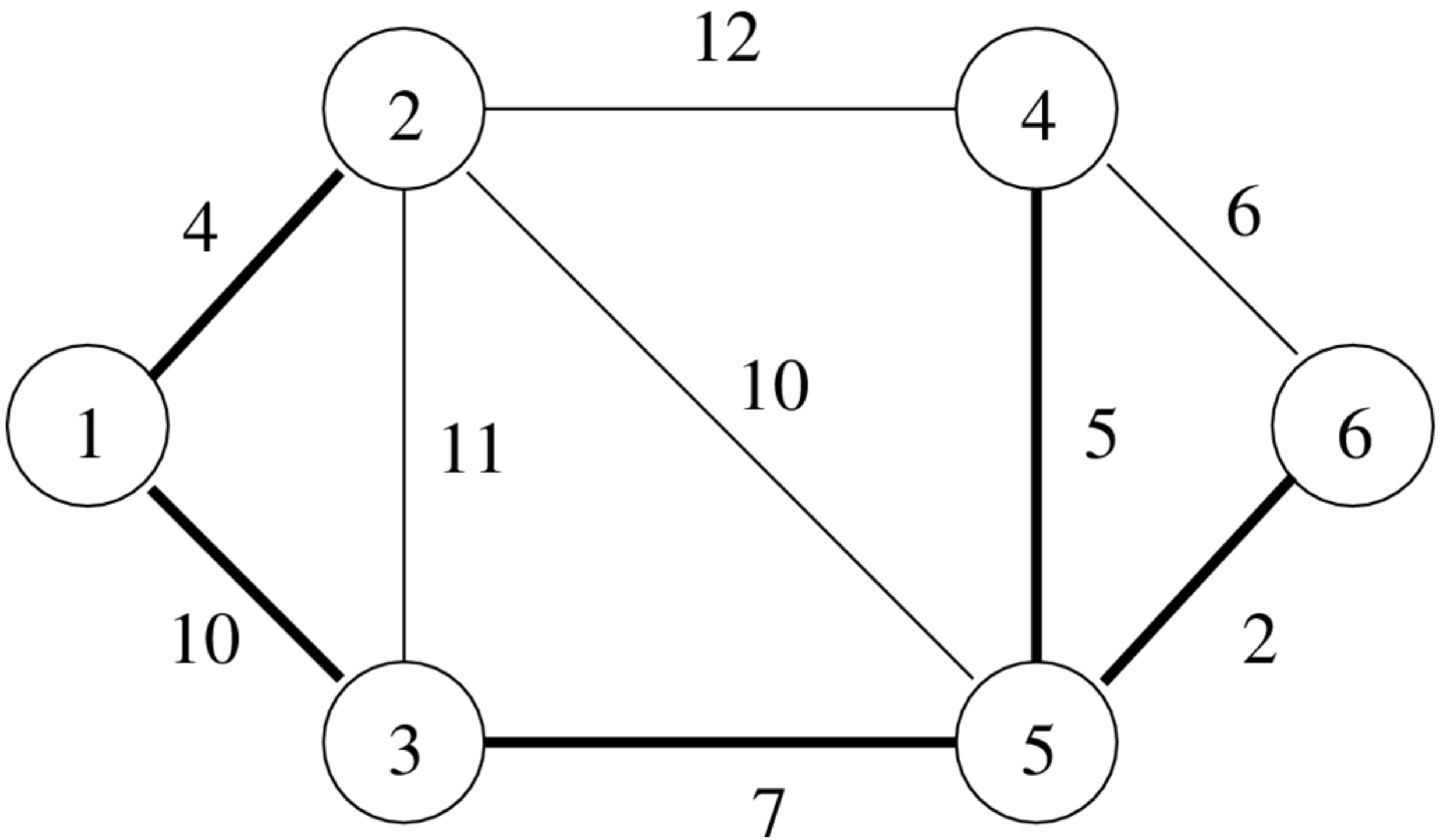
Proof of Theorem 13.1

1. Show if T^* is a MST, then T^* must satisfy the Cut Optimality Conditions.

2. Show if any tree T^* satisfies the Cut Optimality Conditions, then T^* is a MST.

Path Optimality Conditions

Theorem 2. [13.3] *A spanning tree T^* is a MST if and only if for every non-tree arc (k, l) of G, $c_{ij} \leq c_{kl}$ for every arc (i, j) contained in the path in T^* connecting nodes k and l.*



Proof of Theorem 13.3

1. Show if T^* is a MST, then T^* satisfies the Path Optimality Conditions.

2. Show if for every non-tree arc (k, l) of G $c_{ij} \leq c_{kl}$ for every arc (i, j) contained in the path in T^* connecting nodes k and l, then T^* is a MST.

Algorithm Based on Cut Optimality

- Prim's algorithm is motivated by the cut optimality conditions.
- We build up the tree one edge at a time as one connected component.
- In each iteration, we will connect one more node to the current tree.
- We do this by adding the edge that is the minimum length edge across the cut induced by the current set of connected nodes.
- Why does this guarantee optimality?
- How do we do this?

Prim's Algorithm

algorithm *Prim*

$T = \emptyset$

$S = \{1\}$; $\bar{S} = N - \{1\}$

while ($|S| < n$) do

find arc (i, j) in $[S, \bar{S}]$ with minimum cost

$T = T \cup \{(i, j)\}$

$S = S \cup \{j\}$; $\bar{S} = \bar{S} - \{j\}$

Complexity

- Number of iterations?
- Dominant step of each iteration?

Prim's Algorithm

- For each node $j \in \bar{S}$
  - $d(j) =$ min cost of arcs in the cut incident to a node $j \notin \bar{S}$
  - $d(j) = \min\{c_{ij} : (i,j) \in [S, \bar{S}]\}$
  - pred(j) $= i$ such that $c_{ij} = \min\{c_{ij} : (i,j) \in [S, \bar{S}]$.
- To find min cost arc, compute $\min\{d(j) : j \in \bar{S}\}$.
- Suppose $\hat{j}$ is the min, then (pred($\hat{j}$),j) is min cost arc.
- Move $\hat{j}$ to S and update distance and predecessor labels for nodes adjacent to $\hat{j}$.

Running Time of Prim's Algorithm

- Note that Prim's Algorithm is a graph search procedure.
- However, the procedure for determining the search order is more complex than previous ones.
- At each step, we need to update some node labels and then be able to determine the node with the minimum label.
- The key to implementing the procedure is an efficient data structure.
- What is the running time for a naive implementation?

Implementation with Priority Queues

- The running time depends critically on how keep track of the minimum label as the algorithm progresses.
- To get a strongly polynomial time algorithm, we must use a more general data structure for maintaining a *priority queue*.
- For a given order set H, this data structure should support the operations
  - `push(item, value)` (to add and change value of an item)
  - `peek()`
  - `pop()`

Binary Heaps

- A *binary heap* is a balanced binary tree with additional structure that allows it to function efficiently as a priority queue.
- The additional structure needed to support these operations is that each node has a higher priority than either of its children.
- Balanced binary trees can be stored very efficiently in a single array.
  - The root is stored in position 0.
  - The children of the node in position i are stored in positions $2i+1$ and $2i+2$.
  - This determines a unique storage location for every node in the tree and makes it easy to find a node's parent and children.
  - Using an array, basic operations can be performed very efficiently.

Creating the Heap

- Any node whose priority is higher than either of its children is said to satisfy the *heap property*.
- Consider a tree in which all nodes except for the root have the heap property.
- We can easily transform this into a tree in which every node has the heap property (how?).
- This operation is called `heapify()`.
- By calling `heapify()` on each node, starting at the lowest level and working upward, we can transform an unordered binary tree into a heap.
- This is how we create the initial heap.
- Note that this step is unnecessary for implementing Dijkstra's. Why?

Operations on a Heap

- The node with the highest priority is always the root.
- To change the priority of a node
- To insert a node
- To delete a node
- What are the running times of these operations?

Analyzing Prim's with a Binary Heap

Algorithm Based on Path Optimality

- Kruskal's algorithm motivated by path optimality conditions.
- We build up the tree one edge at a time, but this time we build multiple components simultaneously.
- In each step, we will add the minimum edge that does not form a cycle with the edges already added.
- Why does this guarantee optimality?
- How do we implement it?

Kruskal's Algorithm

algorithm *Kruskal*

sort edges in non-decreasing order of length
LIST := $\emptyset$
while ($|$LIST$| < |N| - 1$ and $\exists$ unexamined edges) do
e := unexamined edge with minimum length
if adding e to LIST does not create a cycle
add e to LIST
else discard e

Kruskal's Algorithm: Complexity

- The algorithm has two steps.
  - Sorting the edge list: $O(m \log m) = O(m \log n)$
  - Building the tree: ??
- To determine which edges we are allowed to add in each step requires a data structure for storing *connected components*.
- The data structure must support two operations.
  - `find(i, j)`: Are i and j in the same component?
  - `union(i, j)`: Merge the components i and j.

Quick Find Implementation of Union-Find

- The simplest implementation involves an array of length n.
- We will maintain the array such that two items are in the same subset if and only if the array entries are equal.
- This makes the `find(i, j)` constant time, so we call this implementation *quick find*.
- How do we implement `union(i, j)`?
- What is the running time?
- Note that this could also be implemented using linked lists.

Quick Union Implementation of Union-Find

- To speed up the union operation, we maintain the array in a different fashion.
- We will consider the i^{th} entry of the array to be a pointer to another item.
- To perform `find(i, j)`,
  - Follow the pointers from nodes i and j until reaching a node that points to itself, called the *representative*
  - If the same representative is reached from both nodes i and j, then they are in the same subset.
- To perform `union(i, j)`, perform the find operation and then point the representative for i to the representative for j.
- What is the performance now?

Weighted Quick Union

- Note that the quick union algorithm essentially builds a tree out of the nodes in each component, with the root being the representative.
- As in a heap, the running time of the find operation depends on the depth of the trees.
- Each union operation essentially connects two trees together by pointing the root of one tree to the root of the other.
- One way to limit the depth of the tree is to always point the smaller tree to the larger one.
- This ensures that each find takes less than $\log n$ steps.
- Note that we must now keep track of the number of nodes in each tree, but that's easy to do.
- Another approach is to keep track of the height of each tree and always point the shorter tree to the taller one.

Path Compression

- Ideally, we would like each item to point directly to the representative of its subset.
- One possibility is to simply keep track of all the nodes encountered in the path to the root.
- After reaching the root, set all the nodes on the path to point to the root.
- This is easy to implement recursively and doesn't change the asymptotic running time.
- An easier method to implement is *compression by halving*, which is setting each node to point to its grandparent.
- Combining weighted quick union with path compression yields a total running time for connected components of approximately $O(m)$.

Analyzing Kruskal's with Optimized Union-Find