Graphs and Network Flows
ISE 411

Lecture 3

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References for Today's Lecture

- Required reading
  - Sections 17.2-17.5
- References
  - AMO Sections 2.3
  - CLRS Section 22.1

Designing Algorithms

- In this class, we will discuss the development of algorithms that are both correct and efficient.
- How do we know if an algorithm is correct and what do we mean by efficient?

Proving Correctness

- Correctness of an algorithm must be proven mathematically.
- For the algorithms we'll study, this will not be easy in some cases.
- Many algorithms simply require that the output satisfy some easily verifiable criterion and follow one of the following paradigms.
  - Iterative: The algorthm executes a loop until a termination condition is satisfied.
  - Recursive: Divide the problem into one or smaller instances of the same problem.
- In both cases, we must prove both that the algorithm terminates and that the result is correct by ensuring the criterion is satisfied.
  - Correctness of iterative algorithms is typically proven by showing that there is an *invariant* that holds true after each iteration.
  - Recursive algorithms are almost always proven by an induction argument.

Proving Correctness for Optimization Problems

- Correctness of an optimization algorithm requires more work.
- Typically, we will need to ensure that some particular optimality criteria are achieved.
- Proving that the algorithm terminates will involve showing that a certain progress towards achieving optimality is made on each step.
- These proofs are often not very obvious.

Example: Insertion Sort

A simple algorithm for sorting a list of numbers is insertion sort

```
def insertion_sort(l):
    for i in range(1, len(l)):
        save = l[i]
        j = i
        while j > 0 and l[j - 1] > save:
            l[j] = l[j - 1]
            j -= 1
        l[j] = save
```

Why is this algorithm correct? What is the invariant?

Example: Calculating Fibonacci Numbers

As an example of a recursive algorithm, consider the following function for calculating the n^{th} Fibonacci number.

```
def fibonacci1(n):
    if n == 1 or n == 2:
        return 1
    else:
        return fibonacci1(n-1) + fibonacci1(n-2)
```

- The correctness of this function does not really need to be proven formally, but to illustrate, we could prove it using induction.
- A formal inductive proof requires
  - a base case; and
  - an inductive hypothesis
- What are they in this case?
- How do we know the algorithm terminates?

Algorithm Analysis

- We will be analyzing algorithms from both an *empirical* and a *theoretical* point of view.
- Theoretical analysis involves determining analytically the *worst case* number of basic operations required to execute an algorithms on a given class of graphs.
- Empirical analysis involves real-world performance of a given implementation over a given test set.
- This introduces many (irrelevant) factors that need to be controlled for in some way.
  - Test platform (hardware, language, compiler)
  - Measures of performance (what to compare)
  - Benchmark test set (what instances to test on)
  - Algorithmic parameters
  - Implementational details
- It is much less obvious how to perform a rigorous analysis in the presence of so many factors.
- Practical considerations prevent complete testing.

Measures of Performance

- For the time being, we focus on sequential algorithms.
- What is an appropriate measure of performance?
- What is the goal?
  - Compare two algorithms.
  - Improve the implementation of a single algorithm.
- Possible measures
  - Empirical running time (CPU time, wallclock)
  - Representative operation counts

Measuring Time

- There are three relevant measures of time taken by a process.
  - User time measures the amount of time (number of cycles taken by a process in "user mode."
  - System time the time taken by the kernel executing on behalf of the process.
  - Wallclock time is the total "real" time taken to execute the process.
- Generally speaking, user time is the most relevant, though it ignores some important operations (I/O, etc.).
- Wallclock time should be used cautiously/sparingly, but may be necessary for assessment of parallel codes,

Representative Operation Counts

- In some cases, we may want to count operations, rather than time
  - Identify bottlenecks
  - Counterpart to theoretical analysis
- What operations should we count?
  - Profilers can count function calls and executions of individual lines of code to identify bottlenecks.
  - We may know a priori what operations we want to measure (example: API calls in a graph class).

Test Sets

- It is crucial to choose your test set well.
- The instances must be chosen carefully in order to allow proper conclusions to be drawn.
- We must pay close attention to their size, inherent difficulty, and other important structural properties.
- This is especially important if we are trying to distinguish among multiple algorithms.

Graphs for Testing

- Graph algorithms can perform much differently on graphs with different properties.
- Graphs that arise in applications are not generally "random."
- The most important property is usually *density*, which is the ratio of the number of edges to the number of nodes.
- Most graphs that arise in practice are *sparse*.
- In performing various tests, it will be important to be able to generate *random graph* that preform like graphs that might arise in practice.
- How should we do this?

Generating Random Graphs: Unweighted

- Random edges
- Fixed degree
- Random degree
- k-neighbors
- Euclidean

Generating Random Graphs: Weighted

- Euclidean
- Random

Other Sources of Graphs

- Transaction graph
- Function call graph
- Social graph
- Interval graph
- be Bruijn graph

Comparing Algorithms

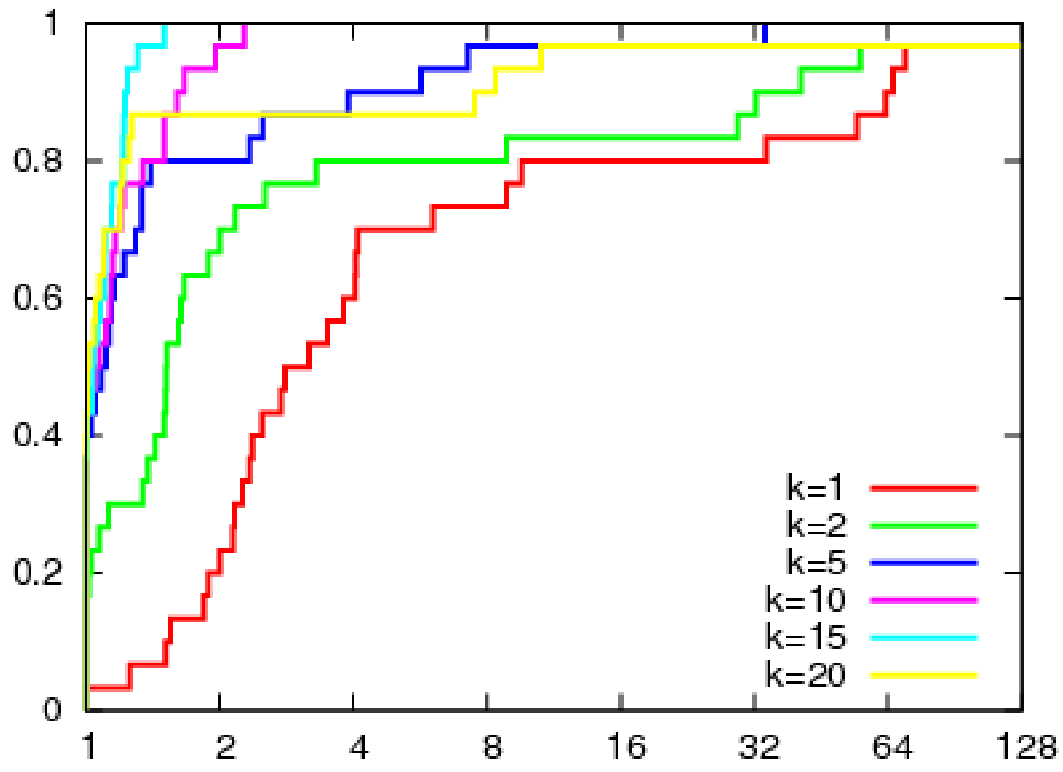
- Given a performance measure and a test set, the question still arises how to decide which algorithm is "better."
- We can do the comparison using some sort of summary statistic.
  - Arithmetic mean
  - Geometric mean
  - Variance
- These statistics hide information useful for comparison.

Accounting for Stochasticity

- In empirical analysis, we must take account of the fact that running times are inherently stochastic.
- If we are measuring wallclock time, this may vary substantially for seemingly identical executions.
- In the case of parallel processing, stochasticity may also arise due to asynchronism (order of operations).
- In such case, multiple identical runs may be used to estimate the affect of this randomness.
- If necessary, statistical analysis may be used to analyze the results, but this is beyond the scope of this course.

Performance Profiles

- Performance profiles allow comparison of algorithms across an entire test set without loss of information.
- They provide a visual summary of how algorithms compare on a performance measure across a test set.



Example: Calculating Fibonacci Numbers

- Let's try to measure how long it takes to calculate the n^{th} Fibonacci number using the recursive algorithm.
- Here is a small routine that returns the execution time of a function for calculating a fibonacci number.

```
def timing(f, n):
    print 'Calculating fibonacci number', n
    start = time()
    f(n)
    return (time()-start)

>>> print timing(fibonacci1, 10)
0.00299978256226
>>> print timing(fibonacci1, 30)
31.0210001469
```

Example: Calculating Fibonacci Numbers (cont'd)

- Notice that we are passing a function as an argument to another function.
- Since functions are just objects, we can put them on lists and pass them as argument, which is very useful.
- What happened with the second function call??
- Why is this function apparently so inefficient??

Calculating Fibonacci Numbers: Second Implementation

- Second Try: Store and reuse intermediate results.

```
def fibonacci2(n):
    f = [0, 1, 1]
    for i in range(3, n+1):
        f.append(f[i-1] + f[i-2])
    return f[n]
```

- Are there any downsides of this implementation?

Calculating Fibonacci Numbers: Third Implementation

- Third Try: Only store the intermediate results that are needed.

```
def fibonacci3(n):
    a, b = 0, 1
    for i in range(n):
        a, b = b, a+b
    return a
```

Calculating Fibonacci Numbers: Comparing Recursive and Iterative

```
def timing(f, n):
    print 'Calculating fibonacci number', n
    start = time()
    f(n)
    return (time()-start)

>>> print timing(fibonacci1, 10)
0.00299978256226
>>> print timing(fibonacci3, 10)
0.0
```

Why is the running time of `fibonacci3` 0?

Calculating Fibonacci Numbers (cont'd)

- The problem with the previous example was that the execution time of the function was so small, it could not be measured accurately.
- To overcome this problem, we can call the function repeatedly in a loop and then take an average.

```
def timing(f, n, iterations = 1):
    print 'Calculating fibonacci number', n

    start = time()
    for i in range(iterations):
        f(n)
    return (time()-start)/iterations

>>> print timing(fibonacci1, 10, 1000)
0.0021319996948
>>> print timing(fibonacci3, 10, 1000)
3.61999988556e-05
```

Example: Calculating Fibonacci Numbers (cont'd)

- Note that the third argument to the function has a default value and is optional in calling the function.
- With this new function, we get a sensible measurement.
- Here, `fibonacci1()` is not obviously inefficient—we do not see this until we try larger numbers.

Running Time as a Function of "Input Size"

- Typically, running times grow as the "amount" of data (number of inputs or magnitude of the inputs) grows.
- We are interested in knowing the general trend.
- Let's do this in the fibonacci case.

```
algos = [fibonacci1, fibonacci3]
symbols = ['bs', 'rs']
symboldict = dict(zip(algos, symbols))
actual = {}
for a in algos:
    actual[a] = []
    for i in range(1, 30):
        actual[a].append(timing(a, i))
# create plots
for a in algos:
    plt.plot(range(1, 30), actual[a], symboldict[a])
plt.show()
```

Plotting the Data

- To plot the data, we use `matplotlib`, a full-featured package that provides graphing capabilities similar to Matlab.
- Plotting the results of the code on the previous slide, we get the following.

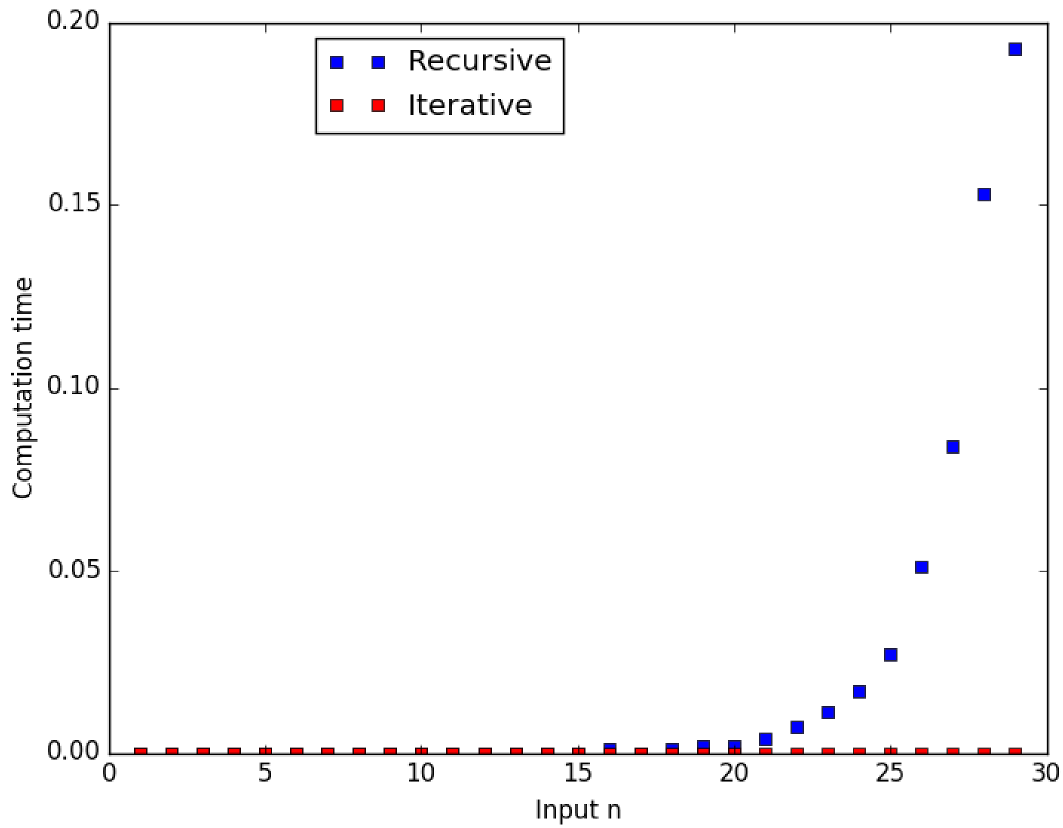


Figure 1: Running times of recurive versus iterative algorithms

Theoretical Analysis

- Can we derive the graph on the previous slide "theoretically"?
- In a basic theoretical analysis, we try to determine how many "steps" would be necessary to complete the algorithm.
- We assume that each "step" takes a constant amount of time, where the constant depends on the hardware.
- We might also be interested in other resources required for the algorithm, such as memory.
- What is the "theoretical" running time for each of the fibonacci algorithms?
  - Aside from the recursive calls, there are only roughly 2 "steps" in each function call.
  - The number of function calls **is** the n^{th} Fibonacci number!

Theortical Analysis

- Let's try to compare our theoretical prediction to the empirical data from earlier.
- What are the "units" of measurement?
- To put the numbers on the same scale, we need to either determine the hardware constant or count the number of "representative operations"

```
theoretical = []
actual = []
n = range(1, 30)
for i in range(1, 30):
    actual.append(timing(a, i))
    theoretical.append(fibonacci(i))
# figure out the constant factor to put times on the same sc
scale = actual[algos[0]][-1]/theoretical[-1]
theoretical = [theoretical[i]*scale for i in range(len(n))]
plt.plot(n, actual, 'bs')
plt.plot(n, theoretical, 'ys')
plt.show()
```

Plotting the Data

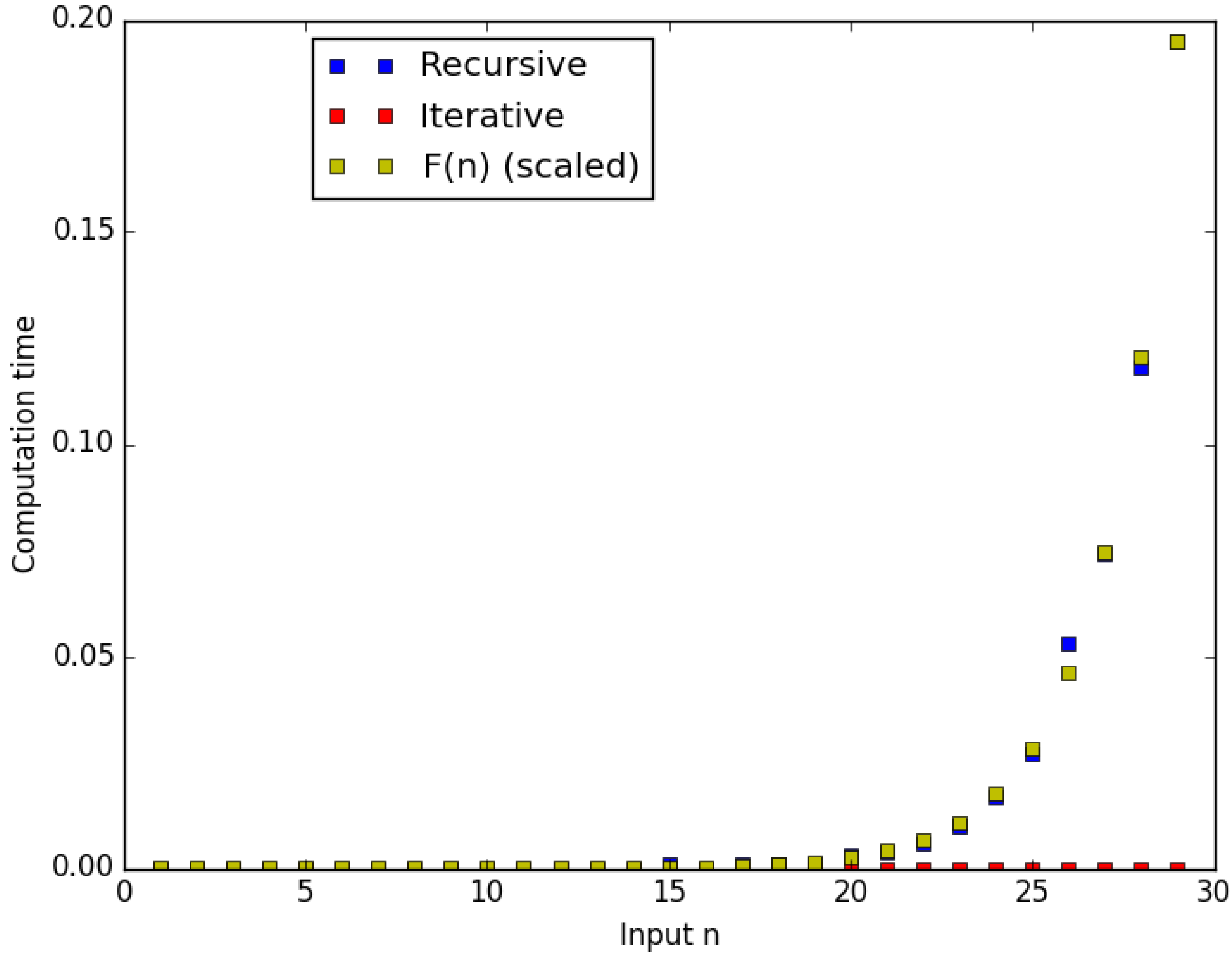


Figure 2: Running times of recurive versus iterative algorithms

More Complex Algorithms

- For most algorithms we will encounter, the analysis is not quite so straightforward.
  - We may not be able to derive the theoretical running time so easily.
  - The algorithm may behave very differently on different inputs.
- What do we want to know?
  - Best-case
  - Worst-case
  - Average-case
- It may depend on how much we know about the instances that will be encountered in practice or how risk-averse we are.

Example: Sorting

- Let's again consider the insertion sort algorithm.
  - How should we test it?
  - How about random instances?
  - Can we guess anything about the algorithm theoretically?

Insertion Sort: Simple Empirical Analysis

Generating random inputs of different sizes, we get the following empirical running time function.

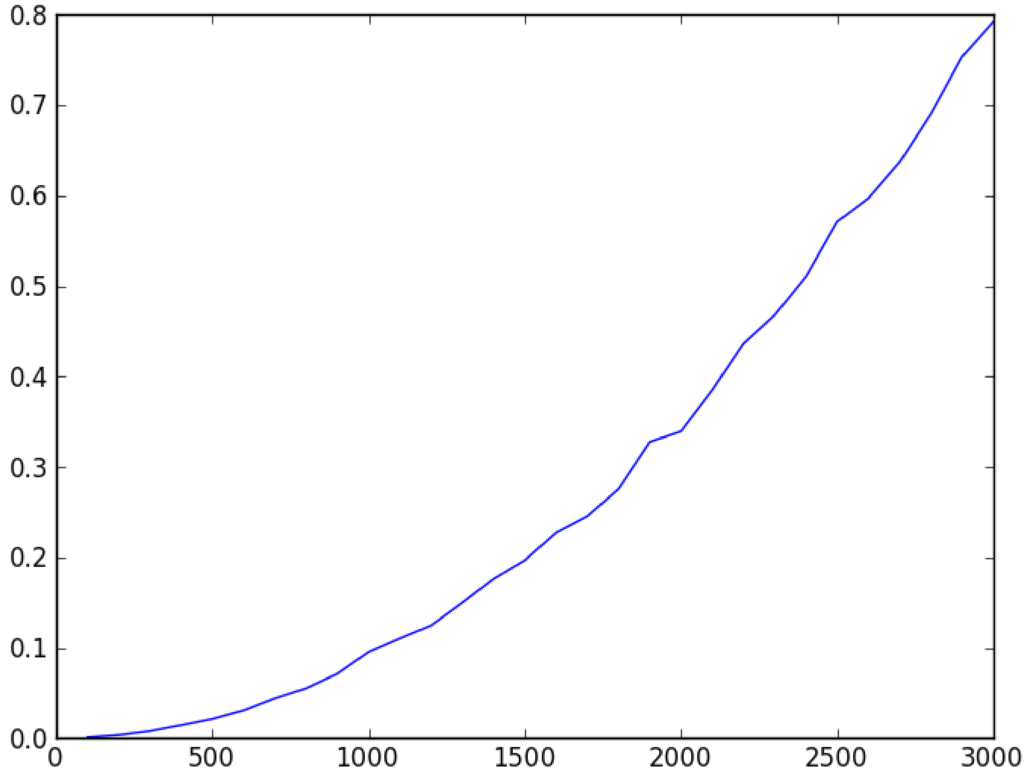


Figure 3: Running time of insertion sort on randomly generated lists

What is your guess as to what function this is?

Insertion Sort: Theoretical Analysis

- What is the maximum number of steps the insertion sort algorithm can take?
- On what kinds of inputs is the worst-case behavior observed?
- What is the "best" case?
- On what kinds of inputs is this best case observed?
- Do you think that empirical analysis will tell us everything we need to know about this algorithm?

Operation Counts

- One way of avoiding the dependence on hardware is to count "representative operations".
- What are the basic operations in a sorting algorithm?
  - Compare
  - Swap
- Most sorting algorithms consist of repetitions of these two basic operations.
- The number of these operations performed is a proxy for the empirical running time that is independent of hardware.

Plotting Operation Counts

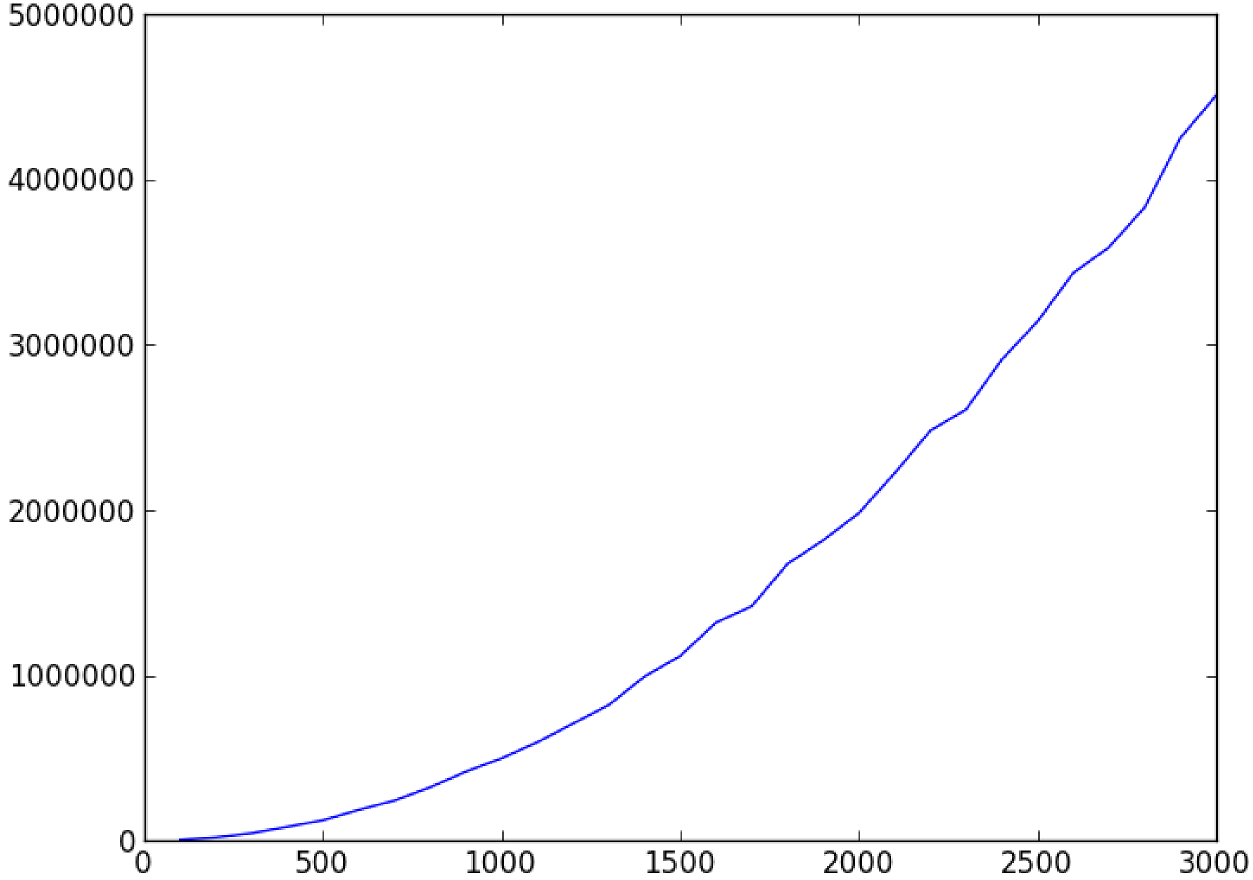


Figure 4: Operation counts for insertion sort on randomly generated lists

Obtaining Operation Counts

- One way to obtain operation counts is using a profiler.
- A profiler counts function calls and all reports the amount of time spent in each function in your program.

```
>>> cProfile.run('insertion_sort_count(aList)', 'cprof.out')
>>> p = pstats.Stats('cprof.out')
>>> p.sort_stats('cumulative').print_stats(10)
```

```
 ncalls  tottime  percall  cumtime  percall function
      1    1.011    1.011    3.815    3.815 insertion_sort
 251040    0.507    0.000    0.507    0.000 shift_right
 252027    0.393    0.000    0.393    0.000 compare
    999    0.002    0.000    0.002    0.000 assign
```

Bottleneck Operations

- If an algorithm is not running as efficiently as we think it should, we may want to know where efforts to improve the algorithm would best be spent.
- Bottleneck analysis breaks an algorithm into parts (mpdules) and analyzes each part using the same analysis as we use for the whole.
- By determine the running times of individual modules, we can determine which part is the most crucial in improving the overall running time.
- To do this, we can make a graph showing the percentage of the running time taken by each module as a function of input size.
- This should make it obvious which module is the bottleneck.
- As we analyze more complex algorithms, we will do some of these kinds of analyses.

From Empirical to Theoretical

- Next, we will look at methods of doing theoretical analysis.
- These are much cleaner methods, in some sense, since many extraneous details of the test environment do not have to be considered.
- For sequential algorithms, theoretical analysis is often good enough for choosing between algorithms.
- It is less ideal with respect to tuning of implementational details.
- For parallel algorithms, theoretical analysis is far more problematic.
- The details not captured by the model of computation can matter much more.
- We will not consider parallel algorithms in this class.