Graphs and Network Flows
IE411

Lecture 20

Dr. Ted Ralphs

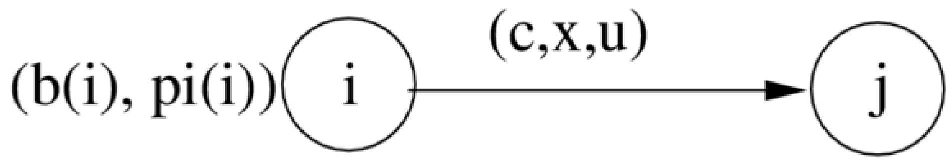
Network Simplex Algorithm

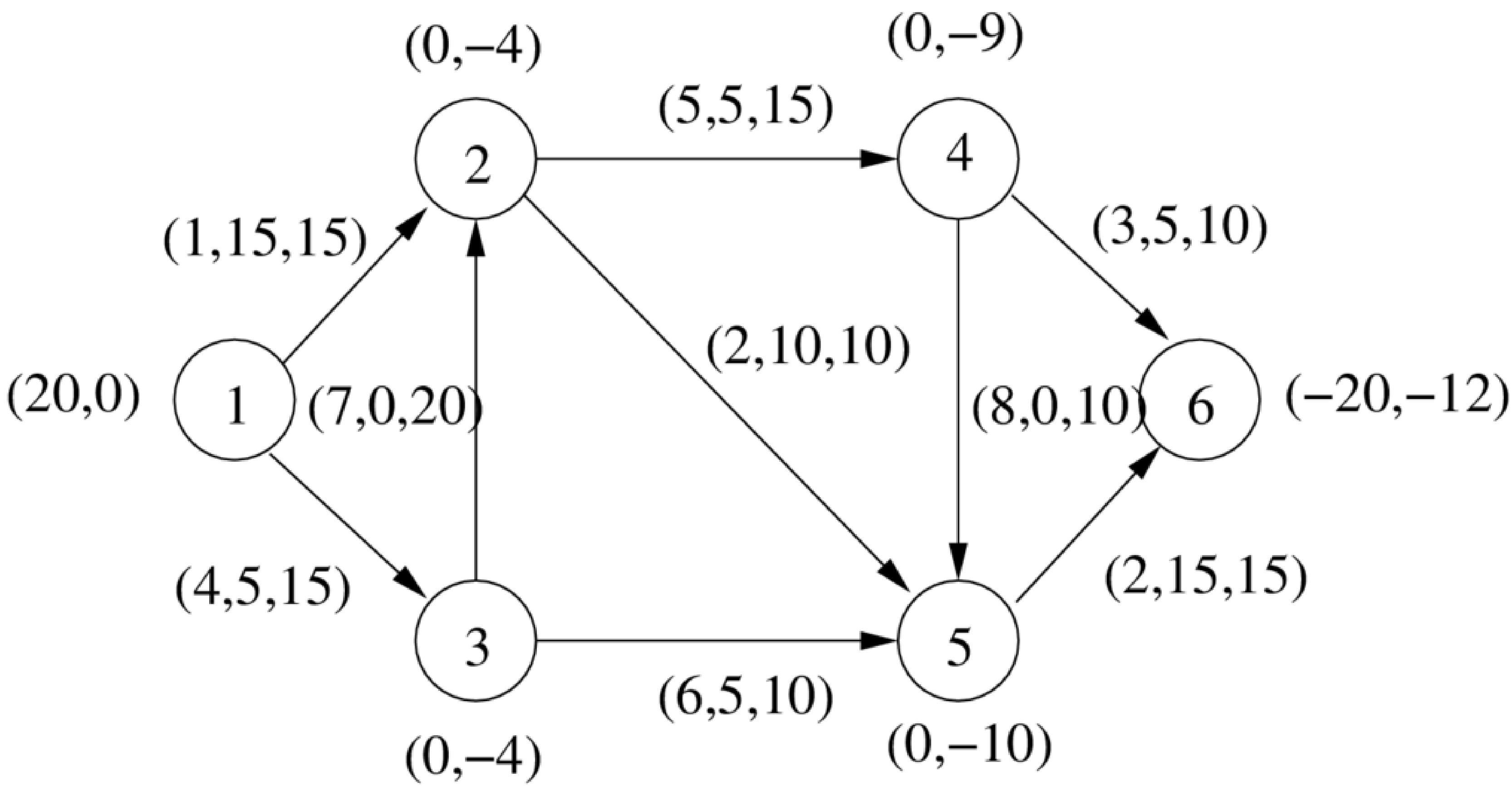
Input: A network $G = (N, A)$, a vector of capacities $u \in \mathbb{Z}^A$, a vector of costs $c \in \mathbb{Z}^A$, and a vector of supplies $b \in \mathbb{Z}^N$

Output: x represents a minimum cost network flow

Determine an initial feasible tree structure (T, L, U)
Let x be flow and π be node potentials associated with (T, L, U)
while Some non-tree arc violates the optimality conditions **do**
  Select an entering arc (k, l) violating optimality conditions
  Add arc (k, l) to tree and determine leaving arc (p, q)
  Perform a tree update and update solutions x and π
end while

Example


(c,x,u)
(b(i), pi(i))
i
j


(0,–4)
(0,–9)
(5,5,15)
2
4
(1,15,15)
(3,5,10)
(2,10,10)
(20,0)
1
(7,0,20)
(8,0,10)
6
(–20,–12)
(4,5,15)
(2,15,15)
3
5
(6,5,10)
(0,–4)
(0,–10)

Degeneracy in Network Simplex

- Network simplex does not necessarily terminate in a finite number of iterations
- Poor choices of entering and leaving arcs lead to *cycling*
- Maintaining a *strongly feasible spanning tree* guarantees finite termination and speeds up the running time
- A pivot iteration is *non-degenerate* if $\delta > 0$ and is *degenerate* if $\delta = 0$
- A degenerate iteration occurs only if T is a degenerate spanning tree.
- If two arcs tie while determining the value of δ, the next spanning tree will be degenerate.

Strongly Feasible Spanning Trees

Let (T, L, U) be a spanning tree structure for a MCFP with integral data. A spanning tree T is *strongly feasible* if

- every tree arc with zero flow is upward pointing (toward root) and every tree arc with flow equal to capacity is downward pointing (away from root)
- we can send a positive amount of flow from any node to the root along the tree path without violating any flow bound.

These two definitions are equivalent. Proof?

Modifications to Network Simplex Algorithm

- Initial Strongly Feasible Spanning Tree
  - Does our construction algorithm work?
    - A non-degenerate spanning tree is always strongly feasible.
    - A degenerate spanning tree is sometimes strongly feasible.
- Leaving Arc Rule
  - Select the leaving arc as the last blocking arc encountered in traversing the pivot cycle W along its orientation starting at the apex w.
  - Proof: Show that next spanning tree is strongly feasible.

Termination

- Each non-degenerate pivot strictly decreases objective function, so number of non-degenerate pivots is finite.
- To show: The pivot rule maintains the invariant that each spanning tree solution is strongly feasible.
  - Consider W_2, the part of the cycle from p to apex: no arc can be blocking by pivot rule.
  - Consider W_1, the part of the cycle from apex to q:
    * If pivot is non-degenerate, then must be able to send flow backwards to root.
    * If pivot is degenerate, then (p, q) must be contained in the part of the cycle from apex to k. Since the previous tree was strongly feasible and flows don't change, we must still be able to send positive flow back along W_1.
- Note that each degenerate pivot must decrease the sum of the node potentials, so the number of denegerate pivots in between each successive non-degenrate pivot must also be finite.

Network Simplex and Simplex for LP

- Network simplex is an implementation of the simplex method for general LPs with upper and lower bounds on the variables.
- Tree solutions correspond to basic solutions in the simplex method.
- To see this, recall from the homework that a directed graph is acyclic if and only if its arc-node incidence matrix is lower triangularizable.
- The number of linearly independent constraints in our formulation of the MCFP is $n - 1$.
- Any basis matrix thus consists of $n - 1$ linearly independent columns.
- It is easy to show that such a basis matrix must have all 1's on the diagonal and must be a tree.

Network Simplex and Simplex for LP (cont.)

- The node potentials are the dual values from the LP and reduced costs are the reduced costs of the arcs.
- Each iteration of network simplex corresponds to a pivot operation in general simplex.
  - Find a nonbasic (nontree) variable (arc) with negative reduced cost fixed at its lower or positive reduced cost fixed at its upper bound.
  - Increase the value of this variable until one of the basic variables hits its bound.
  - Remove the blocking variable from the basis.
- Because of the special form of the problem, we do not need to maintain the basis inverse explicitly.

Dual Network Simplex

- As in general simplex, there is a dual version of the algorithm.
- In this version, we maintain optimality conditions, while trying to achiee feasibility.
- We start with a (possibly infeasible) solution that satisfies optimality conditions and choose a tree arc whose flow violates its bounds.
- This arc is the *leaving* arc.
- We want to push flow around some cycle until the arc reaches its bound.
- The *entering arc* is the one with the “correct” orientation that has the smallest reduced cost (absolute value).
- There is a finite version of this algorithm that uses a perturbation technique similar to that used in general simplex.

Polynomial Algorthms for MCFPs

- As with the maximum flow problem, we can use scaling to reduce the dependence of running time on U and C.
- By scaling the capacitites, we can get a running time of $O(m \log U S(n, m, nC))$.
- By scaling the costs, we can get a running time of $O(n^2 m \log(nC))$.
- By scaling both, we get a running time of $O(nm \log U \log nC)$.
- The minimum mean cycle-canceling algorithm has a strongly polynomial running time of $O(n^2 m^3 \log n)$ (or $O(n^2 n^2 lognC)$).

Sensitivity Analysis

- Determine changes in optimal solution resulting from changes in data
  - arc cost
  - supply/demand
  - arc capacity
- Assuming spanning tree structure remains unchanged, if change in data affects
  - optimality $\rightarrow$ perform primal pivots to achieve optimality
  - feasibility $\rightarrow$ perform dual pivots to achieve feasibility

Cost Sensitivity Analysis

Suppose the cost of arc (p, q) increases by λ units.

Case 1 (p, q) is a non-tree arc

Case 2 (p, q) is a tree arc

Supply/Demand Sensitivity

- Suppose supply/demand $b(k)$ of node k increases by λ units. Then, the supply/demand $b(l)$ of some node l decreases by λ units.
- From the mass balance constraints, we know that we must ship λ units of flow from node k to node l.
- Let P be the unique tree path from node k to node l. And let $\delta = \min\{\delta_{ij} : (i, j) \in P\}$.
- If $\lambda \leq \delta$, then $\cdots$
- If $\lambda > \delta$, then $\cdots$

Capacity Sensitivity Analysis

- Suppose capacity of (p, q) increases by λ units.
- What do we know about previous optimal solution?
- If (p, q) is a tree arc or a non-tree arc at its lower bound
- If (p, q) is a non-tree arc at its upper bound