

Graphs and Network Flows

IE411

Lecture 19

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Network Simplex Algorithm

Input: A network $G = (N, A)$, a vector of capacities $u \in \mathbb{Z}^A$, a vector of costs $c \in \mathbb{Z}^A$, and a vector of supplies $b \in \mathbb{Z}^N$

Output: x represents a minimum cost network flow

Determine an initial feasible tree structure (T, L, U)

Let x be flow and π be node potentials associated with (T, L, U)

while Some non-tree arc violates the optimality conditions **do**

 Select an entering arc (k, l) violating optimality conditions

 Add arc (k, l) to tree and determine leaving arc (p, q)

 Perform a tree update and update solutions x and π

end while

Initial Spanning Tree Structure

- Idea 1: Recall our assumption that network G contains an uncapacitated directed path between every pair of nodes.
 - How can we satisfy this assumption if it is not true?
 - How can we use this to create an initial structure?
- Idea 2: Suppose we do not want to make this assumption. What can we do?

Checking Optimality

- Suppose we have a feasible (T, L, U) and a corresponding set of π .
- What can we conclude if

$$c_{ij}^{\pi} \geq 0 \quad \forall (i, j) \in L \quad \text{and} \quad c_{ij}^{\pi} \leq 0 \quad \forall (i, j) \in U?$$

- What arcs are *eligible* to enter and what is an arc's *violation*?

Pivot Rules

- Dantzig's Pivot Rule: Arc with maximum reduced cost (expensive)
- First Eligible Pivot Rule: First arc with negative reduced cost (cheap)
- Candidate List Pivot Rule: A compromise

Leaving Arc

- What happens when we add an entering arc (k, l) to the current spanning tree?
- The cycle W created is called *a pivot cycle*.
- If $(k, l) \in L$, define orientation in direction of (k, l) ; if $(k, l) \in U$, define orientation in reverse.
- Let \bar{W} denote the set of forward arcs and let \underline{W} denote the set of reverse arcs.
- What happens to cost of solution if we send additional flow around W in direction of orientation?
- How much flow can we augment?

Augmenting Flow in Pivot Cycle

- What happens when we augment flow along W ?
- The maximum flow change δ_{ij} on arc $(i, j) \in W$ satisfies:

$$\delta_{ij} = \begin{cases} u_{ij} - x_{ij} & \text{if } (i, j) \in \bar{W} \\ x_{ij} & \text{if } (i, j) \in \underline{W} \end{cases}$$

- Therefore, to maintain feasibility, $\delta = ?$
- Algorithmically, how do we find the pivot cycle and determine δ ?

Leaving Arc

- What is a *blocking arc*?
- What is a *non-degenerate* iteration?
- What is a *degenerate* iteration?
- If two arcs tie in determining δ , what can we say about the next spanning tree?
- What is the difficult part of this step?

Updating the Tree

At this point, we have identified the leaving arc (p, q) associated with the entering arc (k, l) .

- Can $(p, q) = (k, l)$?
- What do we need to update?
 - Update flow
 - * Find apex and determine δ .
 - * Augment flow and update data structure.
 - Update node potentials.
 - * Deletion of (p, q) creates T_1 (containing root) and T_2 .
 - * Depending on orientation of (p, q) , all node potentials in T_2 change by $\pm c_{kl}^\pi$.

Termination

- If each pivot operation is non-degenerate, it is easy to show that the algorithm terminates finitely.
- If one or more pivot operations is degenerate, ...