Graphs and Network Flows
IE411

Lecture 16

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Outline

- Minimum Cost Flow Problems
  - LP Formulation
  - Optimality Conditions
  - Cycle Canceling Algorithm

LP Formulation

Let $G = (N, A)$ be a directed network with a cost c_{ij} and a capacity u_{ij} associated with every arc $(i, j) \in A$.

Associated with each node $i \in N$ is a number $b(i)$; we refer to node i as a *supply* node if $b(i) > 0$ and as a *demand* node if $b(i) < 0$.

Let x_{ij} denote the amount of flow sent on arc (i, j).

The objective of the *minimum cost flow problem* is to determine a least cost shipment of a commodity through a network in order to satisfy demands at certain nodes from available supplies at other nodes.

LP Formulation

$$\text{Minimize} \quad z(x) = \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b(i) \quad \forall i \in N \tag{2}$$

$$x_{ij} \leq u_{ij} \quad \forall (i,j) \in A \tag{3}$$

$$x_{ij} \geq l_{ij} \quad \forall (i,j) \in A \tag{4}$$

Relationship to Shortest Path and Maximum Flow

- What changes are required to model the shortest path problem?

- What changes are required to model the maximum flow problem?

Assumptions

A1. All data (cost, supply/demand, capacity) are integer.

A2. The network is directed.

A3. The supplies/demands at the nodes satisfy $\sum_{i \in N} b(i) = 0$, and there is a feasible solution.

A4. G contains an uncapacitated directed path between every pair of nodes.

A5. All arc costs are non-negative.

Residual Network $G(x)$

Replace each arc $(i, j) \in A$ by two arcs:
(i, j) with cost c_{ij} and residual capacity $r_{ij} = u_{ij} - x_{ij}$
(j, i) with cost $-c_{ij}$ and residual capacity $r_{ji} = x_{ij}$

Optimality Conditions

- Recall the Shortest Path Optimality Conditions...
- Why are they useful?
  - Provide simple validity check
  - Suggest an algorithm
  - Provide mechanism for establishing correctness of an algorithm

Min Cost Flow Optimality Conditions

We will consider three (equivalent) optimality conditions.

- Negative Cycle Optimality Conditions
- Reduced Cost Optimality Conditions
- Complementary Slackness Optimality Conditions

Negative Cycle Optimality Conditions

Theorem 1. [9.1] *A feasible solution* x^* *is an optimal solution of the minimum cost flow problem if and only if the residual network* $G(x^*)$ *contains no negative cost (directed) cycle.*

Proof:

1. Show that if a feasible solution x^* is an optimal solution of the minimum cost flow problem, then $G(x^*)$ contains no negative cost (directed) cycle.

2. Show that if $G(x^*)$ contains no negative cost (directed) cycle, then x^* is optimal.

Cycle-Canceling Algorithm

- Maintains a feasible solution; attempts to improve objective function value
- Establishes a feasible flow to start
- Iteratively finds a negative cost directed cycle in residual networks and augments flow along cycle
- Terminates when residual network contains no negative cost directed cycle

Generic Cycle-Canceling Algorithm

algorithm *cycle-canceling* (Klein, 1967)
begin
  establish a feasible flow x in the network
  while $G(x)$ contains a negative cycle **do**
    identify a negative cycle W
    $\delta = \min\{r_{ij} : (i,j) \in W\}$
    augment δ units in cycle W and update $G(x)$
end

Example

Complexity of Generic Cycle-Canceling Algorithm

- How many iterations?

- How much work during each iteration?

Implementations of Cycle-Canceling Algorithm

- Negative Cycle with Maximum Improvement
  - $\mathcal{O}(m \text{ log } (mCU))$ – Barahona and Tardos (1989)
  - Note: Max Improvement Cycle is NP-Complete
- Negative Cycle with Minimum Mean Cost
  - Goldberg and Tarjan (1988)
  - mean cost of a cycle $W = (\sum_{(i,j)\in W} c_{ij})/|W|$
  - Identify Minimum Mean Cycle in $\mathcal{O}(nm)$ or $\mathcal{O}(\sqrt{n}\ m\text{log}(nC))$
  - Min Cost Flow $\mathcal{O}(\text{min}\{nm\text{log}(nC), nm^2\text{log}(n)\})$ iterations

Integrality Property

Theorem 2. [9.10] *If all arc capacities and node supplies/demands are integer, then the minimum cost flow problem always has an integer minimum cost flow.*

Proof: