

# Graphs and Network Flows

## ISE 411

### Lecture 1

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## References for Today's Lecture

- Required reading
  - Sections 17.1, 19.1
- References
  - AMO [Chapter 1](#) and [Section 2.1 and 2.2](#)

# Introduction to Graphs and Network Flows

- What is a *graph*?
- What does *network flow* mean?
- The word *network* can be interpreted according to the dictionary definition.
  - electrical networks
  - communication networks
  - transportation networks (highways, railways, airline)
- The word *flow* comes from the movement of something from one point to another.
  - electricity
  - information
  - person or vehicle
  - inventory
  - money
  - Physical goods
- We will rigorously define the word *graph* shortly.

## Graph Problems

- Graphs model the *connectivity relationships* between items in a set.
- Specifically, we will specify that there is a direct link between certain pairs of items.
- In some cases, there will be a direction to the link.
- This will allow us to ask questions such as the following.
  - Is  $x$  connected “directly” to  $y$ ?
  - Is  $x$  connected to  $y$  “indirectly,” i.e., through a sequence of direct connections?
  - What is the set of of all items connected to  $x$ , directly or indirectly?
  - What is the shortest number of connections needed to get from  $x$  to  $y$ ?

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# Applications of Graphs

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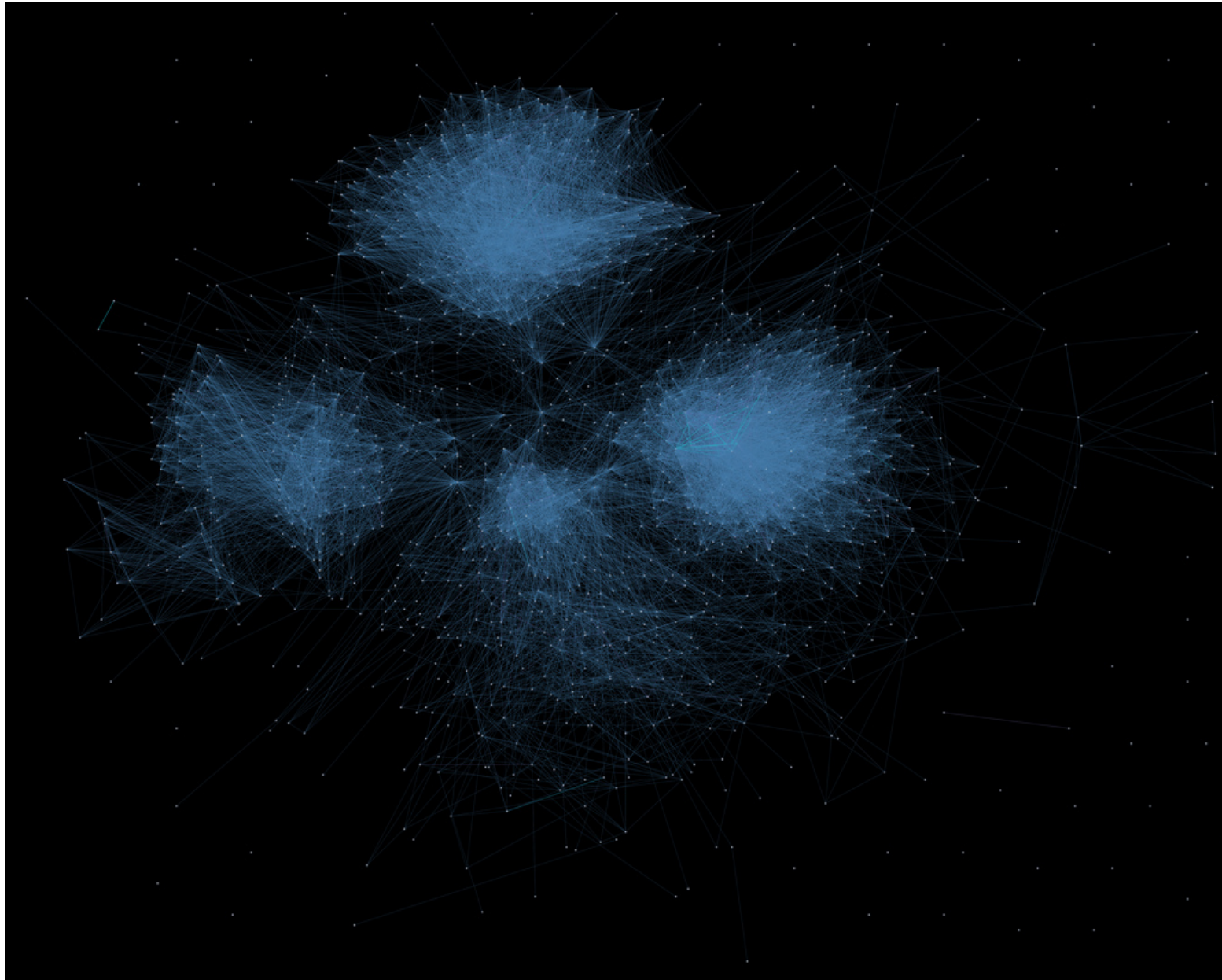
- Maps
- Social Networks
- World Wide Web
- Circuits
- Scheduling
- Communication Networks
- Electricity Networks
- Biological Networks
- Matching and Assignment

# Graphs from Social Networks

LinkedIn Maps Ted Ralphs's Professional Network  
as of August 28, 2012

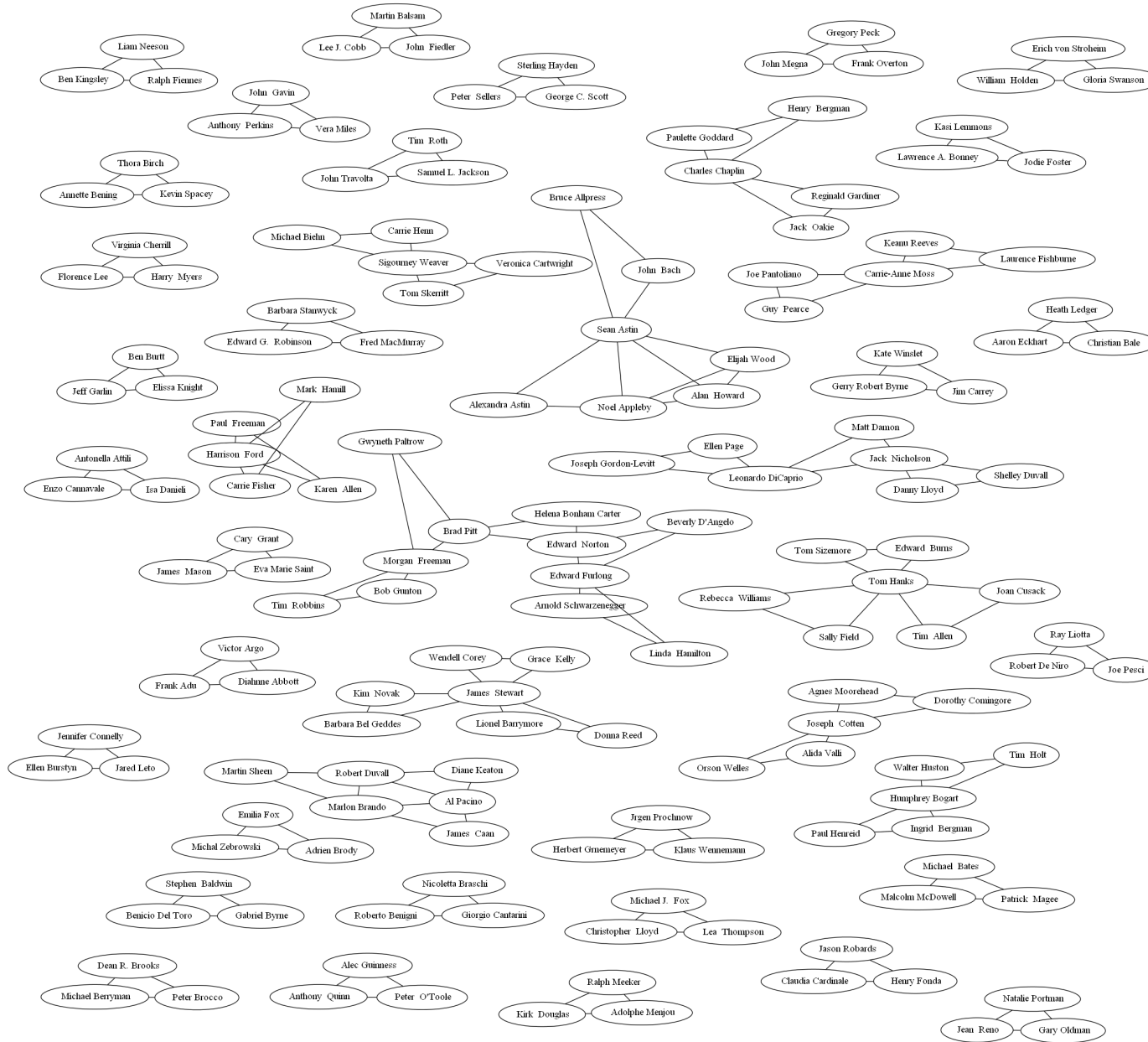


# A Facebook Graph

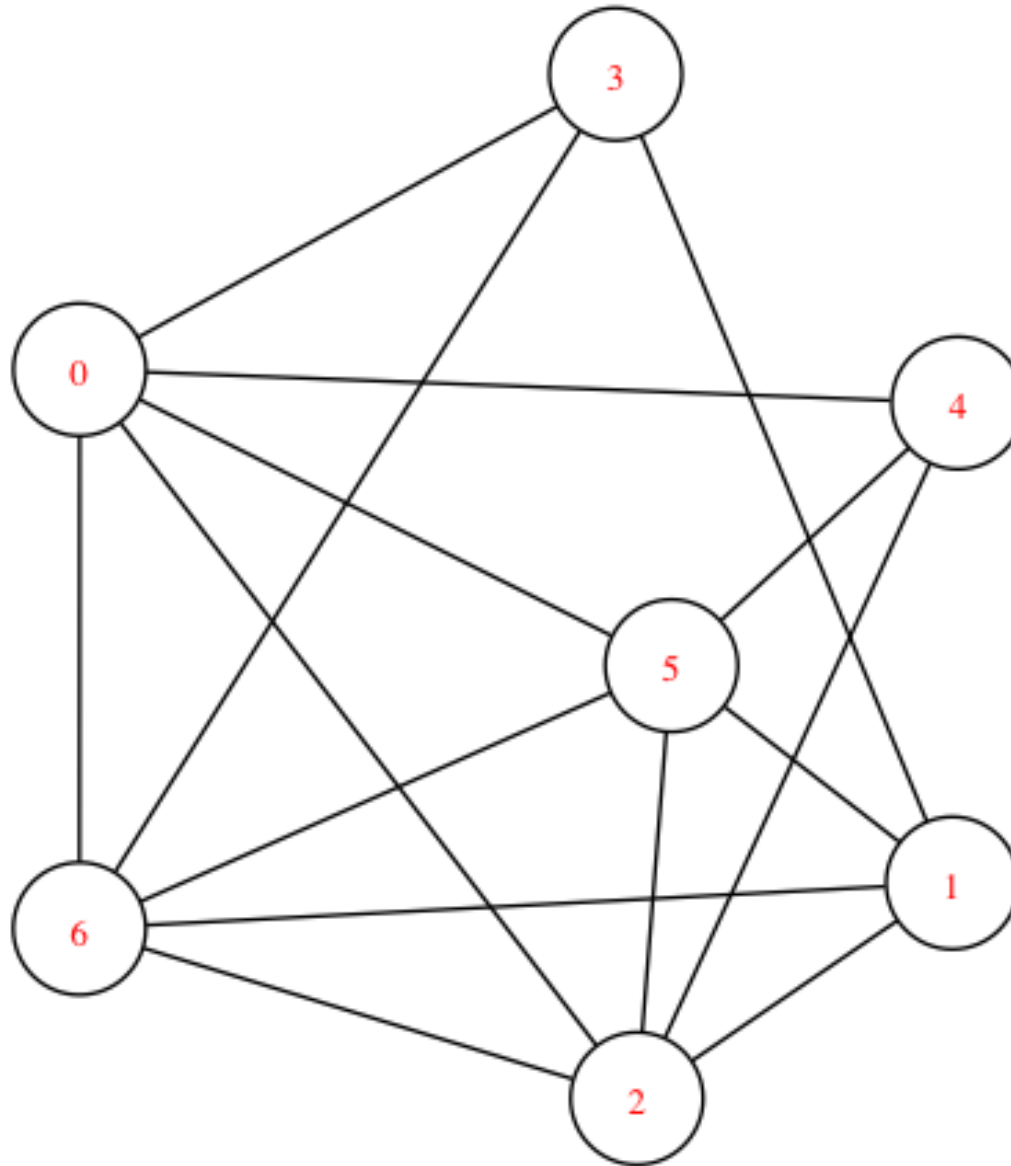




# Graphs for Fun



## Example of an Abstract Graph



# Network Flow Problems

- Network flow problems arise in many, many contexts
  - travel/transportation
  - logistics
  - manufacturing
  - telecommunications
  - chemistry/biology
  - finance
- Much of early work was *descriptive*.
- Our interest is in *prescriptive* models.

## Basic Categories of Network Flow Problems

- Most of the problems we will encounter involve analyzing an existing network.
- Problem types
  - What is the shortest path from point to point?
  - What is the maximum throughput that can be achieved?
  - What is the minimum cost of moving a fixed quantity of goods from point to point?
- For the most part, the flow models we discuss will be *static*.
- We may discuss how to model dynamic flows near the end of the course.

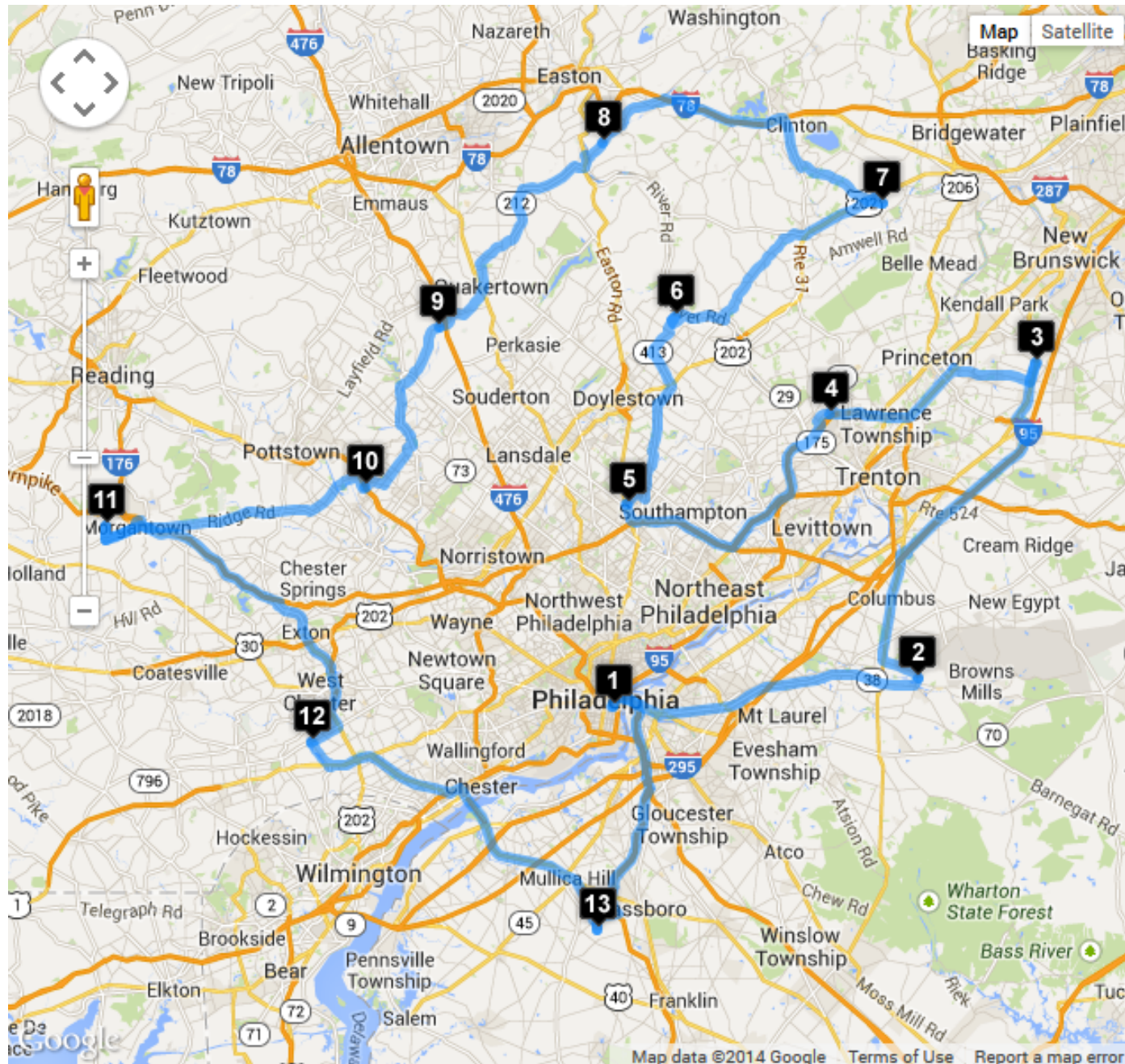
## How Hard Are They?

- There is a well-defined sense in which flow problems are among the “easiest” optimization problems of practical importance.
- They are much easier to solve than general linear programs, for example.
- So why do we want to spend a whole course studying them?
- The applications of flow models require algorithms to be *extremely fast*.
  - GPS
  - Internet packet routing
- Algorithms for network flow problems are also embedded in a wide range of techniques for solving higher level models.
- For this reason, we will spend a substantial part of the course discussing **data structures** and other implementation issues.

# Graphs

- A *graph* is an abstract object used to model connectivity relations.
- A *graph* consists of a list of items, along with a set of connections between the items.
- The study of such graphs and their properties, called *graph theory*, is hundreds of years old.
- Graphs can be visualized easily by creating a physical manifestation.
- There are several variations on this theme.
  - The connections in the graph may or may not have an *orientation* or a *direction*.
  - We may not allow more than one connection between a pair of items.
  - We may not allow an item to be connected to itself.
- For now, we consider graphs that are
  - *undirected*, i.e., the connections do not have an orientation, and
  - *simple*, i.e., we allow only one connection between each pair of items and no connections from an item to itself.

# A Tour of a Graph Made from Map Data



## Graph Terminology and Notation

- In an undirected graph, the “items” are usually called *vertices* (sometimes also called *nodes*).
- The set of vertices is denoted  $V$  and the vertices are indexed from 0 to  $n - 1$ , where  $n = |V|$ .
- The connections between the vertices are *unordered pairs* called *edges*.
- The set of edges is denoted  $E$  and  $m = |E| \leq n(n - 1)/2$ .
- An undirected graph  $G = (V, E)$  is then composed of a set of vertices  $V$  and a set of edges  $E \subseteq V \times V$ .
- If  $e = \{i, j\} \in E$ , then
  - $i$  and  $j$  are called the *endpoints* of  $e$ ,
  - $e$  is said to be *incident* to  $i$  and  $j$ , and
  - $i$  and  $j$  are said to be *adjacent* vertices.



## More Terminology

- Let  $G = (V, E)$  be an undirected graph.
- A *subgraph* of  $G$  is a graph composed of an edge set  $E' \subseteq E$  along with all incident vertices.
- A subset  $V'$  of  $V$ , along with all incident edges is called an *induced subgraph*.
- A *path* in  $G$  is a sequence of vertices such that each vertex is adjacent to the vertex preceding it in the sequence.
- A path is *simple* if no vertex occurs more than once in the sequence.
- A *cycle* is a path that is simple except that the first and last vertices are the same.
- A *tour* is a cycle that includes all the vertices.

## Directed Graphs

- A directed graph  $G = (N, A)$  consists of a set of nodes  $N$  and a set of arcs  $A$  whose elements are ordered pairs of nodes.
- In a directed graph, ordering is important. Consider an arc  $(i, j)$ .
  - Node  $i$  is the *tail* and node  $j$  is the *head*.
  - We say  $(i, j)$  is incident to nodes  $i$  and  $j$ .
- Sometimes it will be convenient to refer to the number of nodes  $(|N|)$  as  $n$  and the number of arcs  $(|A|)$  as  $m$ .
- How do we change the definition to define an *undirected* graph?

## Networks and Subgraphs

- A network is a (directed) graph whose nodes and/or arcs have associated numerical values.
  - costs
  - capacities
  - supplies or demands
- The graph  $G' = (N', A')$  is a subgraph of  $G = (N, A)$  if  $N' \subseteq N$  and  $A' \subseteq A$ .
  - We say that  $G' = (N', A')$  is the subgraph *induced* by  $N'$  if  $A'$  contains each arc of  $A$  with both endpoints in  $N'$ , i.e.,  $A' = A \cap (N' \times N')$ .
  - In the example graph, if  $N' = \{1, 2, 3, 5\}$ , what is the subgraph induced by  $N'$ ?

## Degree in Directed Graphs

- The degree of a node is the number of arcs to which it is incident.
  - The indegree of node  $i$ ,  $I(i)$ , is the number of incoming arcs.
  - The outdegree of node  $i$ ,  $O(i)$ , is the number of outgoing arcs.
  - $\sum_{i \in N} I(i) = \sum_{i \in N} O(i) = ?$

Node	Degree	InDegree	OutDegree
1	2	0	2
2	3	1	2
3	4	2	2
4	3	2	1
5	2	2	0

## Directed Paths

- A path in a directed graph  $G = (N, A)$  is a path in the *underlying undirected graph* (the graph obtained by ignoring the orientations of the arcs).
- A *directed* path requires the correct orientation of the arcs.

## Directed Cycles

- As with paths, a cycle is cycle in the underlying undirected graph.
- In the example graph,  $1 - 2 - 3 - 1$  is a cycle.
- A directed cycle is a cycle obeying the orientation of the arcs.
- The example graph has no directed cycle. What do we call such graphs?

## Other Concepts

- What does it mean for a graph to be *connected*?
- What about *strongly connected*?
- What is a *cut*?
- What is an *s-t cut*?

# Trees

- A *tree* is a connected, undirected graph that contains no cycle.
- Elementary properties of trees
  - A tree on  $n$  nodes has exactly  $n - 1$  arcs.
  - A tree has at least 2 leaf nodes.
  - Every pair of nodes is connected by a unique path.
- What is a *subtree*?
- What is a *forest*?
- What are the directed analogues?



## Application Example (Ahuja et al., 1.7)

- Have you ever thought about how word processing programs decide where to break lines?
- In *TeX*, an optimization procedure is used to decompose the words of a paragraph into lines.
- The objective of the optimization problem is to maximize the attractiveness of the paragraph.
- Suppose that a paragraph consists of  $n$  words and each word is assigned a sequence number from  $1$  to  $n$ . Let  $c_{ij}$  denote the attractiveness of a line if it begins with the word  $i$  and ends with the word  $j - 1$ .

## Shortest Path Problem

- Consider a directed network  $G = (N, A)$ .
  - Each arc  $(i, j) \in A$  has an associated cost/length  $c_{ij}$ .
  - There is one distinguished node  $s$  called the *source node*.
- Define the *length* of a directed path as the sum of the lengths of the arcs in the path.
- The objective of the shortest path problem is to determine for every node  $i \in N \setminus \{s\}$  a shortest length directed path from  $s$  to  $i$ .

## Paragraph Problem as a Network Flow Problem

- Given the values of the  $c_{ij}$ 's, formulate a shortest path problem to decompose the paragraph into lines in such a way as to maximize the total attractiveness.

## Reformulation and Modeling

- Given the values of the  $c_{ij}$ 's, construct an appropriate network and formulate an optimization problem on that network to decompose the paragraph into lines in such a way as to maximize the total attractiveness.
- General procedure for reformulation
  - Decide what nodes and arcs represent conceptually and what other network data is needed.
  - Decide what type of network problem the original problem represents.
  - Construct the network from the problem input data.
  - Solve the network problem.
  - Translate the solution to the network problem back into a solution to the original problem.
- What is the model and problem in this case?