Problem Set 5
IE411 Graphs and Network Flows
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1. Let x^1 and x^2 be two distinct (alternate) optimal solutions to a given minimum cost flow problem. Suppose that for some arc (k,l), $x^1_{kl} = p$ and $x^2_{kl} = q$ with $p < q$. Show that for every $0 \leq \lambda \leq 1$, there is an optimal solution x (possibly not integral) with $x_{kl} = \lambda p + (1-\lambda)q$.

2. Let x^* be an optimal solution of the minimum cost flow problem. Consider the admissible network G^o utilized in the primal-dual algorithm. Show that the minimum cost flow problem has an alternative optimal solution if and only if G^o contains a directed cycle.

3. The *constrained minimum cost flow problem* is a minimum cost flow problem with an additional *budget constraint* of the form

$$\sum_{(i,j)\in A} d_{ij} x_{ij} \leq D \tag{1}$$

   for $d \in \mathbb{Z}^A$ and $D \in \mathbb{Z}$.

   (a) Show that the constrained minimum cost flow problem need not have an integer solution, even when all the data are integer.

   (b) For the constrained minimum cost flow problem, we say that a solution x is an *augmented tree solution* if some partition of the arc set into the subsets $T \cup \{(p,q)\}$, L, and U satisfies the following two properties:

   - T is a spanning tree, and
   - by setting $x_{ij} = 0$ for each arc (i,j) in L and $x_{ij} = u_{ij}$ for each arc in U, we obtain a unique flow on arcs in $T \cup \{(p,q)\}$ that satisfies the mass balance constraints and the budget constraint.

   Show that the constrained minimum cost flow problem always has an optimal solution that is an augmented tree solution.

4. The balanced assignmnet problem is another variation of the classical assignment problem. Given n people and n tasks, let c_{ij} denote the amount of work it would take person i to do job j. In the balanced version of the problem, we are interested in choosing a pairing of workers to jobs that distributes the work as evenly as possible. Formally, the objective is to minimize the difference between the most costly and least costly assignment. Our goal is to develop an algorithm for solving this problem.

   Suppose that we sort the arc costs and let $c_1 < c_2 < \cdots < c_k$ be the sorted list of the distinct values of these costs ($k \leq m$). Let $FS(l,u,M)$ denote a subroutine that take as input two numbers l and u such that $1 \leq l \leq u \leq k$ and determines whether in some assignment M every arc cost is between c_l and c_u. If no such assignment exists, then we set m to be the empty set.

   (a) Describe an $O(\sqrt{n}\, m)$ algorithm for implementing the subroutine $FS(l,u,M)$.

   (b) Show how to solve the balanced assignment problem by calling the subroutine $FS(l,u,M)$ $O(k)$ times.

5. Let $G = (N, A)$ be an undirected network with a capacity u_{ij} associated with every arc $(i, j) \in A$. For any spannning tree T of G, we define its *capacity* as $\min\{u_{ij} \mid (i, j) \in T\}$ and for any cut Q of G, we define its *value* as the $\max\{u_{ij} \mid (i, j) \in Q\}$.

   (a) Show that the capacity of any spanning tree is a lower bound on the value of every cut (Weak duality).

   (b) Show that the maximum capacity of any spanning equals the minimum value of any cut (Strong duality).