Quiz 2 Sample Questions
IE406 – Introduction to Mathematical Programming
Dr. Ralphs

These questions are from previous years and should you give you some idea of what to expect on Quiz 2.

1. WTH Industries is a producer of two products, Whosamajigits and Watchamacallits. To produce these products, they have available to them three types of workers: underqualified, qualified, and overqualified. The hourly wages of each class of workers and the number of each type of product they can produce per hour is shown below.

	Underqualified	Qualified	Overqualified
Hourly Wage	\$4	\$6	\$8
Whosamajigits/hour	2	3	3
Watchamacallits/hour	0	3	2

Note that each worker may produce the specified quantity of both products simultaneously each hour, so that the interpretation of the above table is that an unqualified worker can produce 2 Whosamajigits per hour, while a qualified worker can produce both 3 Whosamajigits and 3 Watchamacallits during each hour of work. Also note that despite their qualifications, the overqualified workers are lazy and actually slightly less productive than the qualified ones. The production level needed to meet U.S. demand for Whosamajigits is 21 units per hour, while the required production level for Watchamacallits is 15 units per hour.

(a) (5 points) Formulate a linear program to determine the optimal number of each type of worker for WTH to hire in order to minimize wage costs subject to meeting production targets. You can ignore the fact that the solution should be integral in order to avoid human rights violations. Be sure to explain your variables, constraints, and objective.

(b) (5 points) Take the dual of the linear program from part (a).

(c) (10 points) Solve the linear program from part (a) using an appropriate version of the simplex algorithm (do not use a graphical method). Explain your choice of method and show each tableau.

(d) (10 points) Suppose a third product, Whatsamajiggers, is to be added to the product line with an anticipated production level of 24 units per hour. Unqualified workers can produce 1 Whatsamajigger an hour in addition to the 2 Whosamajigits they are already capable of producing, while qualified workers can produce an additional 3 Whatsamajiggers per hour. Overqualified workers have a union contract that does not allow them to produce any Whatsamajiggers. Find the new optimal solution using the simplex algorithm starting from the previous optimal basis.

(e) (5 points) WTH is considering contracting with a foreign distributor to sell Watchamacallits in other countries. What is the minimum price they should be willing to accept for producing additional units?

(f) (5 points) Argue that after the introduction of Whatsamajiggers, overqualified workers can never be hired, even if they agree to a lower wage.

2. Consider the following parametric LP:

$$\begin{aligned} g(\theta) = \min\ & \theta x_1 - x_2 \\ s.t.\quad & x_1 + x_2 \leq 2, \\ & x_1 - x_2 \leq 0, \\ & x_1 \leq 1, \text{ and} \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) (10 points) Use the parametric simplex method to determine $g(\theta)$ for all values of θ.

(b) (10 points) Which extreme points of the feasible region are optimal for *some* value of θ? For those that are not, use a geometric argument to explain why.

3. Consider the linear program $\max_{x \in \mathcal{P}} c^T x$ where $\mathcal{P}$ is a polyhedron with extreme points x_1, x_2, x_3, and x_4, and extreme directions d_1, d_2, and d_3 such that

$$c^T x_1 = 5, \tag{1}$$
$$c^T x_2 = 7, \tag{2}$$
$$c^T x_3 = 4, \tag{3}$$
$$c^T x_4 = 7, \tag{4}$$
$$c^T d_1 = 0, \tag{5}$$
$$c^T d_2 = -3, \text{ and} \tag{6}$$
$$c^T d_3 = 0. \tag{7}$$

(a) (10 points) Characterize the set of all alternative optimal solutions to this problem.

(b) (10 points) Explain why the optimal dual solution must be degenerate.

4. Consider the following linear program:

$$\begin{aligned} max\ & 3x_1 + 3x_2 + 21x_3 \\ s.t.\ & 6x_1 + 9x_2 + 25x_3 \leq 25 \\ & 3x_1 + 2x_2 + 25x_3 \leq 20 \\ & x_1,\ x_2,\ x_3,\ \geq 0 \end{aligned}$$

(a) Determine an optimal basis matrix using a graphical method (hint: consider the dual problem). DO NOT USE THE SIMPLEX METHOD.

(b) Calculate the corresponding primal and dual solutions and show directly that they are optimal.

(c) Determine graphically the set of all right hand sides for which the current basis remains optimal.

5. Consider the following linear program:

$$\begin{array}{lllll} min & 2x_1 & +3x_2 & +x_3 & \\ s.t. & x_1 & +2x_2 & +x_3 & \geq 10 \\ & -x_1 & +x_2 & -2x_3 & \geq 4 \\ & x_1, & x_2, & x_3, & \geq 0 \end{array}$$

(a) Consider the basis matrix formed by taking x_2 and x_3 to be the set of basic variables. Calculate the corresponding basic primal and dual solutions and show that the basis is optimal.

(b) Write down the dual problem and show that the dual solution found in part (a) is a basic feasible solution for the dual program.

(c) Consider simultaneously changing the objective function coefficients of x_2 and x_3 by δ_2 and δ_3 respectively. Derive a system of inequalities that δ_2 and δ_3 should satisfy in order for the current basis to remain optimal.

(d) Consider a column-generation algorithm in which a new variable x_4 with column $[a_{14}, a_{24}]^\top$ in the constraint matrix and objective function coefficient c_4 is to be added to the problem. Determine the objective function for the column-generation subproblem and give an example of a column whose addition to the problem would lower the objective function value if $c_4 = 10$.

6. Consider the LP

$$\begin{aligned} min\ & c^\top x \\ s.t.\ \ & Ax = b \\ & x \geq 0 \end{aligned}$$

in standard form defined by matrix $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Now consider an invertible matrix $D \in \mathbb{R}^{m \times m}$ and define a modified LP in standard form by the matrix DA and right hand side vector Db, i.e., the new LP is

$$\begin{aligned} min \quad & c^\top x \\ s.t.\ \ & (DA)x = Db \\ & x \geq 0 \end{aligned}$$

Note that the constraints of the modified LP are just linear combinations of the constraints of the original LP.

(a) Show that the feasible region for the modified problem is the same as that for the original problem and show by example that this isn't necessarily true if D is not invertible.

(b) For a given set of basic variables with associated basis matrix B in the original problem, give a formula in terms of B for computing the basis matrix $\bar{B}$ corresponding to the same set of basic variables in the modified problem.

(c) For a given set of basic variables, give a formula for computing the corresponding basic primal and dual solutions for the modified problem directly from those for the original problem.

(d) Show that the objective function values of both the primal and dual solutions corresponding to a given set of basic variables are the same in both problems. (Thus, a given basis is optimal for the modified problem if and only if it is optimal for the original problem).

7. Consider an LP $\min\{c^\top x | Ax = b, x \geq 0\}$ in standard form.

   (a) Assuming we can find an initial primal feasible basis, the simplex method will terminate by either finding an optimal solution or detecting that the optimal cost is unbounded. In the case where unboundedness is detected, explain how to use the information in the tableau to exhibit an extreme ray d satisfying the conditions of Theorem 4.14, thereby proving directly that the primal problem is unbounded.

   (b) Assuming we can find an initial dual feasible basis, the dual simplex algorithm will terminate by either finding an optimal solution or detecting that the dual problem is unbounded, i.e., that the primal problem is infeasible. In the case where dual unboundedness is detected, explain how to use the information in the tableau to exhibit a vector p satisfying the conditions of the Theorem 4.6(b), thereby proving directly that the primal problem is infeasible.

8. Consider the LP

$$\begin{aligned} &min\ (c + \theta d)^\top x \\ &s.t. \qquad\quad Ax = (b + \theta f) \\ &\qquad\qquad\quad x \geq 0 \end{aligned}$$

   (a) Suppose $b = c = 0$. Show that if some basis is optimal for $\theta = 1$, then it is optimal for any $\theta \geq 0$.

   (b) For general b and c, show that if some basis is optimal for $\theta = -10$ and $\theta = 10$, then it is optimal for $\theta = 5$.

9. Consider the following linear programming problem and its optimal final tableau shown below.

$$\begin{aligned} \min\ & -2x_1 - x_2 + x_3 \\ \text{s.t.}\quad & x_1 + 2x_2 + x_3 \le 12 \\ & -x_1 + x_2 - 2x_3 \le 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Final tableau:

x_1	x_2	x_3	x_4	x_5	
0	3	3	2	0	24
1	2	1	1	0	12
0	3	-1	1	1	15

(a) Write the dual program and determine its optimal solution by examining the tableau.

(b) Using sensitivity analysis, determine the range of values for the objective function coefficient of x_2 for which the basis shown above remains optimal and determine the optimal tableau if the coefficient of x_2 is changed from -1 to -6.

(c) Determine the range of values of the right hand side of the first constraint for which the basis shown above remains optimal.

(d) Suppose that after obtaining the optimal solution depicted in the final tableau above, it was revealed that the following set of constraints were left out and must also be satisfied:

$$\begin{aligned} 2x_1 + 3x_2 &\le 20 \\ x_1 - x_2 + x_3 &\le 11 \\ 2x_1 - 3x_3 &\le 23 \end{aligned}$$

Use constraint generation to obtain an optimal solution after augmenting the original LP with these three new constraints. (Hint: This only requires a few calculations.)

10. Consider a polyhedron $\mathcal{P}$ with the following extreme points and extreme rays:

$$\begin{aligned} x^1 &= (0,3) && (8) \\ x^2 &= (0,4) && (9) \\ x^3 &= (1,1) && (10) \\ x^4 &= (3,0) && (11) \\ w^1 &= (1,1) && (12) \\ w^2 &= (1,0) && (13) \\ & && (14) \end{aligned}$$

(a) Determine $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ such that $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \geq b\}$.

(b) Describe the polyhedron containing all $c \in \mathbb{R}^n$ for which $\min_{x\in\mathcal{P}} c^\top x > -\infty$.

(c) Determine explicitly the piecewise linear function

$$f(\theta) = \min_{x\in\mathcal{P}} \theta x_1 + x_2.$$

In other words, determine the breakpoints and the slope in each interval. This can be done graphically.

11. The following questions are to be answered TRUE or FALSE. You will be graded primarily on the justification of your answer, not the answer itself.

(a) Deleting a constraint that is binding at optimality will result in a strict decrease in the optimal primal objective function value for a minimization LP.

(b) When changing a single objective function coefficient, if the new coefficient is out of the allowable range for the optimal basis, then the current solution will become suboptimal after the change.

(c) If an LP is unbounded, it is possible to change the right hand side to make the solution finite.

(d) If the variables of an LP all have finite upper and lower bounds, then the LP is infeasible if and only if the dual LP is also infeasible.

12. The following tableau corresponds to an **optimal** basis for a linear programming problem, where x_1, x_2, and x_3 are the original primal variables, and s_1 and s_2 are the slack variables corresponding to the two constraints.

x_1	x_2	x_3	s_1	s_2	
0	8	0	3	2	35
0	1	1	$\frac{1}{2}$	0	$\frac{5}{2}$
1	-2	0	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{5}{2}$

(a) Write the original linear program.

(b) Determine the set of right-hand sides for which this linear program is feasible.

(c) Determine the optimal solution value as a function of the right-hand side of the first constraint as it is varied over all values for which the linear program is feasible.

13. This problem examines the use of the dual simplex algorithm to solve a minimization LP. The dual simplex algorithm can be viewed as a *dual ascent* method because it modifies the dual solution in a prescribed manner at each step in order to achieve an increase in the dual objective function value until an optimal solution is reached. Using an improving search paradigm, the dual solution is modified by determining

an improving feasible direction (for the dual) and moving in that direction as far as possible.

Consider the following linear program and the sequence of tableaux obtained when solving it using the dual simplex algorithm. Note that the rows are presented as vectors of integers along with a scale factor by which that row is multiplied in the tableau. This is done to make the numbers easier to read. PLEASE DON'T FORGET ABOUT THE SCALE FACTOR.

$$
\begin{aligned}
\min\ 3x_1 + 4x_2 + 6x_3 + 7x_4 & \\
2x_1 - x_2 + x_3 + 6x_4 &\geq 6 \\
x_1 + x_2 + 2x_3 + x_4 &\geq 3 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Initial tableau:

Scale factor	x_1	x_2	x_3	x_4	s_1	s_2	RHS
1	3	4	6	7	0	0	0
1	-2	1	-1	-6	1	0	-6
1	-1	-1	-2	-1	0	1	-3

Tableau after first pivot:

Scale factor	x_1	x_2	x_3	x_4	s_1	s_2	RHS
$\frac{1}{6}$	4	31	29	0	7	0	-42
$\frac{1}{6}$	2	-1	1	6	-1	0	6
$\frac{1}{6}$	-4	-7	-11	0	-1	6	-12

Tableau after second pivot (optimal):

Scale factor	x_1	x_2	x_3	x_4	s_1	s_2	RHS
1	0	4	3	0	1	1	-9
$\frac{1}{4}$	0	-3	-3	4	-1	2	0
$\frac{1}{4}$	4	7	11	0	1	-6	12

(a) Graphically depict the dual polyhedron and show the path followed by the dual simplex algorithm.

(b) Explain how to determine the improving direction for the dual solution by examining the tableau.

(c) Determine the sequence of dual solutions and the improving direction for the dual solution followed in each iteration.

(d) Is the optimal *primal solution* shown in the final tableau unique? How can you tell by looking at the final tableau?

(e) Is the optimal *basis* shown in the final tableau unique? How can you tell by looking at the final tableau?

(f) Are any of the dual constraints redundant? How would you interpret this in terms of the primal problem?

14. Consider again the linear program from problem 13.

    (a) Determine the range of objective function coefficients for x_1 for which the current basis remains optimal.

    (b) Formulate a linear program that will determine the largest total perturbation that can be made to the right hand side such that the optimal basis depicted in the final tableau remains feasible, i.e., formulate an LP that determines δ_1 and δ_2 such that the optimal basis remains feasible when the right hand side is changed to $(6 + \delta_1, 3 + \delta_2)^\top$ and such that $|\delta_1| + |\delta_2|$ is maximized.

15. Consider a linear program $\min\{c^\top x \mid a^\top x = b, x \geq 0\}$ in standard form with a single constraint, where $a, c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ (i.e., $m = 1$).

    (a) (10 points) Use linear programing duality to determine an explicit form for optimal primal and dual solutions. (Hint: show that your solutions have the same objective function value)

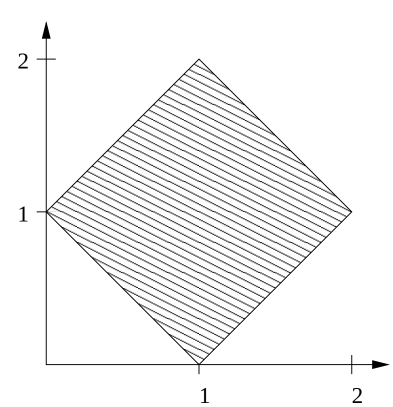
    (b) (15 points) What is the form of the function $F(d) = \min\{c^\top x \mid a^\top x = d\}$? Give an explicit example of this function for specific c, a, and b.

16. Consider a linear program $\min\{c^\top x \mid x \in \mathcal{P}\}$, where $\mathcal{P} \subseteq \mathbb{R}^n$ is a give polyhedron and $c \in \mathbb{R}^n$ is a given vector.

    (a) (15 points) Argue that the set of improving feasible directions at an extreme point $\hat{x}$ of $\mathcal{P}$ is a polyhedral cone.

    (b) (15 points) Argue that the extreme rays of the cone of improving directions from part (a) at a nondegenerate BFS $\hat{x}$ of $\mathcal{P}$ correspond to the basic directions at $\hat{x}$ that would be obtained using the primal simplex algorithm.

17. Consider the following polyhedron $\mathcal{P}$ in two dimensions.



(a) (15 points) What is the form of the function $G(c) = \min\{c^\top x \mid x \in \mathcal{P}\}$ for the polyhedron $\mathcal{P}$ above. Specify the function explicitly.

(b) (15 points) If $c \in \mathbb{R}^n$ is such that the point $(2, 1)$ is an optimal primal solution to $\min\{c^\top x \mid x \in \mathcal{P}\}$ for the polyhedron $\mathcal{P}$ above, what is an optimal dual solution in terms of c?

(c) (15 points) Under what conditions are the primal and dual solutions in part (b) unique?