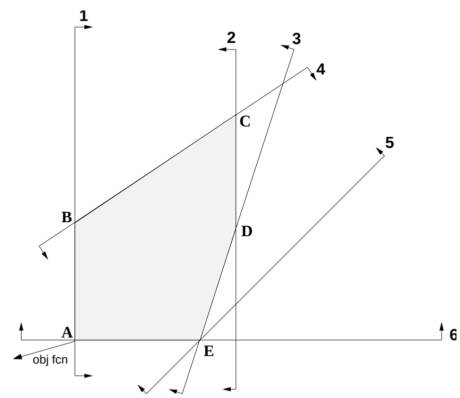
Quiz 1 Sample Questions
IE406 – Introduction to Mathematical Programming
Dr. Ralphs

These questions are from previous years and should you give you some idea of what to expect on Quiz 1.

1. Consider the linear program pictured here, where the feasible region is the shaded area, the objective function vector is the vector pictured with its tail at the point labeled A, and the goal is to MINIMIZE.



   The numbers indicate the indices of the variables corresponding to each (nonnegativity) constraint when the program is expressed in standard form, while the letters indicate the extremal elements of the feasible region. The following questions refer to solution of this linear program by the primal simplex method.

   (a) (10 points) What is the optimal solution to this linear program? Argue geometrically why the solution must be optimal. Is the solution unique? Why or why not?

   (b) (10 points) List all sequences of BFS's that can occur when solving this linear program with the simplex method. For this part, you SHOULD NOT assume that the rule for selecting the variable to enter the basis is to select the one with the most negative reduced cost. In the case of a degenerate pivot, a single BFS may be listed more than once in the list. Indicate the sequences that involve the smallest and largest number of simplex iterations. Justify your answer.

   (c) (10 points) Of the above sequences, which ones can occur if the rule for selecting the variable to enter the basis is to select the variable with the most negative reduced cost (assuming the basic direction vectors all have the same norm)? Justify your answer.

   (d) (10 points) Which path occurs if the tiebreaking rule for selecting the leaving variable in each iteration is to select the variable with the smallest subscript? With the largest subscript? Justify your answer.

2. For the questions below, provide either a short argument supporting the validity of the statement or a specific counterexample. All questions refer to the primal simplex

algorithm applied to a linear program in standard form with feasible region $\mathcal{P}$ and a constraint matrix of full rank.

(a) (10 points) If the only allowable pivots are degenerate, then the current basis is optimal.

(b) (10 points) If the current solution is degenerate, then the objective function value will remain unchanged after the next pivot.

(c) (10 points) If there is a tie in the ratio test, then the next BFS will be degenerate.

(d) (10 points) The existence of a redundant constraint implies the existence of a degenerate BFS.

3. (a) (10 points) Consider a linear program in standard form. An alternative to the method of finding an initial BFS discussed in class is to introduce a single artificial variable with index $n+1$ such that $A_{n+1} = b$. In the modified LP, the solution $\hat{x} \in \mathbf{R}^{n+1}$ such that $\hat{x}_{n+1} = 1$ and $\hat{x}_i = 0$ for all $i \in 1..n$ is then feasible. Describe a method of finding an initial BFS by introducing such an artificial variable in the case, assuming all of the original constraints are inequalities. Be specific.

   (b) (10 points) Use the method described above to find a BFS for the following LP and set up an initial tableau for the simplex method:

$$
\begin{aligned}
\min \quad & x_1 + 3x_2 + 4x_3 \\
s.t. \quad & -x_1 + 2x_2 + x_3 \geq 2, \\
& x_1 - x_2 - 2x_3 \geq 1, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

4. Consider the following simple linear program:

$$
\begin{aligned}
\min \quad & -2x_1 - x_2 \\
s.t. \quad & x_1 + x_2 \leq 2, \\
& x_1 - x_2 \leq 0, \\
& x_1 \leq 1, \text{ and} \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

(a) (5 points) Draw the feasible region and identify all basic feasible solutions. Indicate which solutions are degenerate, which are nondegenerate, and which are optimal. Explain your answer.

(b) (5 points) Write the problem in standard form and set up the initial tableau using the origin as the initial basic feasible solution. What is the initial set of binding constraints (in terms of the original problem)?

(c) (5 points) Do one simplex pivot, choosing the variable with most negative reduced cost to enter the basis. Explain graphically the effect of this pivot. What is the new basic feasible solution and what is the new set of binding constraints? Explain your answer.

(d) (5 points) On the next pivot, you should have only one choice for entering variable, but two choices for leaving variable. Explain this graphically. What will be the new basic feasible solution and the new set of binding constraints in each case? Explain your answer. You do not have to actually perform either of the pivots.

(e) (5 points) Explain graphically which of the two choices of leaving variable in part (d) will lead to an optimal basis. Perform this pivot and show that the new basis is optimal.

(f) (5 points) Determine graphically all bases associated with the optimal basic feasible solution. List the set of basic variables in each. Which are optimal and which are not? Explain your reasoning.

5. The following questions are short answer. Please justify your answers completely.

(a) (10 points) In a simplex tableau for a minimization LP, suppose $\bar{c_j} = -10$ and that variable j is chosen to enter the basis. What is the change in objective function value that occurs after the change of basis if the minimum ratio is 2? Explain your answer.

(b) (10 points) Consider the following optimal tableau, obtained by solving a given linear program.

x_1	x_2	x_3	x_4	x_5	x_6	
0	0	0	0	2	3	1
1	0	2	-1	-1	1	1
0	1	-2	1	2	3	1

By inspection, determine the set of all optimal bases. Explain your answer.

(c) (10 points) Consider the linear program

$$\min_{x \in \mathcal{P}} c^\top x, \tag{1}$$

where $\mathcal{P} = \{x \in \mathbf{R}^n \mid Ax \geq b\}$ is a polyhedron described in inequality form by $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. Show that if $\hat{x} \in \mathcal{P}$ and $A\hat{x} > b$, then $\hat{x}$ is optimal if and only if $c = 0$.

6. Consider a *pre-emptive bi-objective linear program* (PBLP), in which the goal is to find, among all optimal solutions to a given linear program with an associated primary objective function vector, a solution whose cost is minimum with respect to a secondary objective function vector. In other words, consider the linear program

$$\min_{x \in \mathcal{P}} c^\top x, \tag{2}$$

where $\mathcal{P}$ is a given polyhedron described in standard form and $c \in \mathbf{R}^n$ is the primary objective function vector. Let F be the set of all optimal solutions to this linear program. The PBLP with secondary objective function vector $d \in \mathbf{R}^n$ is to determine

$$\min_{x \in F} d^\top x. \tag{3}$$

(a) (10 points) Show that the optimization problem (3) is a linear program.

(b) (10 points) Argue that if there is an optimal solution to (3), then there is an optimal solution that is an extreme point of $\mathcal{P}$.

7. Consider the following linear program:

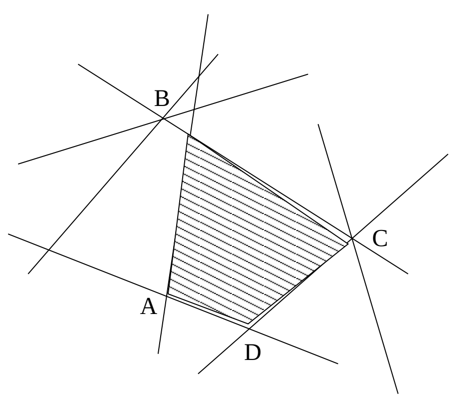
$$
\begin{aligned}
min \quad & x_1 + x_2 + x_3 \\
s.t. \quad & x_1 + 2x_2 + 6x_3 \leq 2 \\
& 2x_1 + 4x_2 + 2x_3 \leq 2 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

(a) (5 points) Write this problem in standard form (the remaining parts refer to the problem in standard form).

(b) (5 points) Is it possible for both x_1 and x_2 to both be basic at the same time? Explain.

(c) (5 points) How many basic solutions are there to this problem? Explain.

8. (a) (5 points) In the two-dimensional polyhedron pictured below, identify the points corresponding to degenerate basic solutions and say why they are degenerate.



(b) (5 points) How can you tell from looking at a tableau if the current solution is degenerate? Explain.

(c) (5 points) Once the variable to enter the basis has been chosen, how can you tell if the pivot will be degenerate, i.e., will lead to a new basis representing the same basic feasible solution? Explain.

9. WiGgy, Inc. is a company whose two products are widgets and gadgets. The demand for widgets and gadgets is practically unlimited, but the company is constrained by its production capacity and cash flow. The factory contains only one kind of machine, which can manufacture either widgets or gadgets. It takes 3 hours to manufacture a widget and 4 hours to manufacture a gadget, and there is a total of 120,000 hours of machine time available in the factory per quarter. The raw materials must be purchased in advance each quarter at a cost of \$3 per widget and \$2 per gadget.

These raw materials must be purchased with existing capital reserves, which total \$90,000 dollars. The selling price of a widget is \$6 and a gadget \$5.

(a) (5 points) Assuming that raw materials are the only cost associated with manufacturing widgets and gadgets, formulate a linear program to maximize profits for the current quarter (Hint: Use units that make the numbers easy to work with).

(b) (10 points) Solve this LP using the tableau method. Briefly explain your steps.

(c) (10 points) Graph the associated polyhedron, show the objective function, and indicate the optimal solution.

(d) (5 points) If the number of available machine hours goes up by 2,000 hours, explain how to obtain the new optimal solution without re-solving the problem from scratch.

(e) <u>Extra Credit</u> (5 points): Assuming the increase in machine hours described above is due to the purchase of new machines, what is the most WiGgy, Inc. should be willing to pay for them?

10. Consider the following linear program:

$$\begin{array}{llllll} min & 10x_1 & +x_2 & & & \\ s.t. & x_1 & +x_2 & +4x_3 & +2x_4 & = 2 \\ & 2x_1 & +6x_2 & & -2x_4 & = 12 \\ & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

(a) (5 points) Set up a tableau for the first phase of the two-phase simplex method.

(b) (15 points) Solve this LP using the two-phase simplex method.

11. Consider again the LP of the Top Brass example from lecture, restated below.

$$\begin{array}{llll} max & 12x_1 & +9x_2 & \\ s.t. & x_1 & & \leq 1000 \\ & & x_2 & \leq 1500 \\ & x_1 & +x_2 & \leq 1750 \\ & 4x_1 & +2x_2 & \leq 4800 \\ & x_1, & x_2 & \geq 0 \end{array}$$

Let y be the basic feasible solution $(250, 1500)$.

(a) (5 points) Determine the set of all feasible directions at y. Justify your answer.

(b) (5 points) Determine the set of all improving directions at y. Justify your answer.

(c) (5 points) Using your results from (a) and (b), show that y is not optimal.

(d) (5 points) Determine the set of all objective functions for which y would be optimal. Justify your answer.

12. Consider the LP

$$
\begin{array}{llllll}
min & -10x_1 & -12x_2 & -12x_3 & & \\
s.t. & x_1 & +2x_2 & +2x_3 & \leq & 20 \\
 & 2x_1 & +x_2 & +2x_3 & \leq & 20 \\
 & 2x_1 & +2x_2 & +x_3 & \leq & 20 \\
 & x_1, & x_2, & x_3 & \geq & 0
\end{array}
$$

and a corresponding optimal tableau

	x_1	x_2	x_3	s_1	s_2	s_3
136	0	0	0	$3\frac{3}{5}$	$1\frac{3}{5}$	$1\frac{3}{5}$
4	0	0	1	$\frac{2}{5}$	$\frac{2}{5}$	$-\frac{3}{5}$
4	1	0	0	$-\frac{3}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
4	0	1	0	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{2}{5}$

(a) (5 points) Determine the basic feasible solution shown in the above tableau and its corresponding basis inverse (B^{-1}).

(b) (10 points) Suppose a variable x_4 is added to the original LP, resulting in the new LP

$$
\begin{array}{lllllll}
min & -10x_1 & -12x_2 & -12x_3 & +\alpha x_4 & & \\
s.t. & x_1 & +2x_2 & +2x_3 & +x_4 & \leq & 20 \\
 & 2x_1 & +x_2 & +2x_3 & +x_4 & \leq & 20 \\
 & 2x_1 & +2x_2 & +x_3 & +x_4 & \leq & 20 \\
 & x_1, & x_2, & x_3, & x_4 & \geq & 0.
\end{array}
$$

Construct the tableau corresponding to the basis from part (a) for this new LP by adding a column for the variable x_4. The reduced cost of x_4 should be expressed as a function of α.

(c) (5 points) Determine for what values of α the current solution remains optimal for the new LP.

13. (10 points) One way in which the simplex algorithm can be used to solve maximization problems is to simply negate the objective function. Explain how the simplex method could be adapted to solve maximization problems directly, i.e., without negating the objective function.

14. The following linear programming model describes a problem of allocating two resources to the annual production of three commodities by a manufacturing firm. The amounts of the three products to be produced are denoted by x_1, x_2, and x_3. The objective function represents the profit associated with each of these products. **Note that this is a maximization problem**.

$$\begin{aligned} \max\ & 10x_1 + 15x_2 + 5x_3 \\ s.t.\quad & 3x_1 + 3x_2 + x_3 \leq 9000 \\ & x_1 + 2x_2 + 2x_3 \leq 4000 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) (10 points) **Without using the simplex method**, verify that there is an optimal solution in which only the first two products are produced. Calculate such an optimal solution. Show your work.

(b) (10 points) Make use of the information derived in (a) to write the corresponding optimal tableau.

(c) (10 points) The R&D department proposes a new product whose production coefficients are represented by $[4, 1]^\top$ (i.e., this would be the new column to be added to the constraint matrix) and whose associated profit 12 per unit. Should this product be produced? Justify your answer.

(d) (5 points) What is the minimal profit for the new product that would result in an optimal solution in which the new product would be produced?

15. The starting and current tableaux of a linear programming problem are given below. Find the values of the unknowns a through l. Show your calculations and provide brief explanations.

Starting Tableau

x_1	x_2	x_3	x_4	x_5	RHS
a	-1	3	0	0	0
b	c	d	1	0	6
-1	2	e	0	1	1

Current Tableau

x_1	x_2	x_3	x_4	x_5	RHS
0	$-\frac{1}{3}$	j	k	l	2
g	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	f
h	i	$-\frac{1}{3}$	$\frac{1}{3}$	1	3

List values here (1 point for each correct value):

$a =$	$b =$	$c =$	$d =$
$e =$	$f =$	$g =$	$h =$
$i =$	$j =$	$k =$	$l =$

16. Consider a linear programming problem P in standard form:

$$\begin{aligned} \min\ & c^\top x \\ s.t.\quad & Ax = b \\ & x \geq 0 \end{aligned}$$

with $A \in \mathbf{R}^{m\times n}$, $c \in \mathbf{R}^n$, and $b \in \mathbf{R}^m$.

(a) (10 points) Explain the intuition behind the following statement: "P is unbounded if and only if there exists a nonzero real vector $d \geq 0$ such that $Ad = 0$ and $c^\top d < 0$." You do not have to write a formal proof.

(b) (10 points) Explain how this relates to the condition that demonstrates unboundedness when performing the simplex algorithm, i.e., that no component of the vector u is positive when performing the ratio test (step 3 in the description of the tableau method on page 100 of the textbook).

(c) (5 points) Graphically depict an unbounded linear programming problem and show a vector d that demonstrates that satisfies the criteria of part (a), i.e., shows that the LP is unbounded. Explain your answer.

17. Consider the following linear program:

$$\begin{aligned} \min\ & c_1x_1 + c_2x_2 \\ s.t.\quad & x_1 + x_2 \leq b_1 \\ & -x_1 + 2x_2 \leq b_2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) (10 points) Write this problem in standard form, and determine the basis inverse corresponding to the basic solution for which $x_1 = 0$ and the first constraint is binding.

(b) (10 points) Using the basis inverse, determine the set of values of b_1 and b_2 for which the corresponding basic solution is feasible.

(c) (10 points) Assuming that $b_1 = 1$ and $b_2 = 2$, use the basis inverse to determine the set of values of c_1 and c_2 such that the corresponding basic feasible solution is optimal.

(d) (10 points) Assuming that $b_1 = 1$, $b_2 = 2$, $c_1 = 1$, and $c_2 = 1$, set up the simplex tableau corresponding to this basic feasible solution and solve it using the simplex algorithm. Explain each step.

18. We say that a constraint with coefficient vector $a \in \mathbf{R}^n$ and right-hand side vector $b \in \mathbf{R}$ is *redundant* for $\mathcal{P}$ if the corresponding half-space includes $\mathcal{P}$, i.e., $\mathcal{P} \subseteq \{x \in \mathbf{R}^n \mid a^\top x \geq b\}$.

    (a) (10 points) Formulate an LP that will determine if a given constraint is redundant for a polyhedron $\mathcal{P}$.

    (b) (10 points) Suppose that while solving an LP, the current tableau shows that the slack variable corresponding to the i^{th} constraint is basic in row j, but every other entry in row j is nonpositive (except for the entry in the column corresponding to the slack variable and the right-hand side entry). Show algebraically that in this case, the i^{th} constraint is redundant and that the column corresponding to the slack variable, along with the j^{th} row of the tableau can be removed and the simplex algorithm continued.

    (c) (10 points) Generalize the result in part (b) to show that for any variable i that is basic in row j of the tableau, if all entries in row j are negative (other than in column j and the right-hand side), then column i and row j can be removed from the tableau and the simplex algorithm continued. Discuss how to how to determine the value of the removed variable after the optimal solution to the reduced problem has been obtained.