Sample Final Examination Questions
IE406 – Introduction to Mathematical Programming
Dr. Ralphs

1. Consider the following linear programming problem and its optimal final tableau.

$$\begin{aligned} \min\ & -2x_1 - x_2 + x_3 \\ & x_1 + 2x_2 + x_3 \le 8 \\ & -x_1 + x_2 - 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

x_1	x_2	x_3	s_1	s_2	
0	3	3	2	0	16
1	2	1	1	0	8
0	3	-1	1	1	12

The parts to this problem are all independent. All parts start from the tableau above, not from the tableau attained in any previous part.

(a) Find the new optimal solution if the coefficient of x_2 is changed from -1 to -6 without resolving from scratch.

(b) Find the new optimal solution if the coefficient of x_2 in the first constraint is changed from 2 to $\frac{1}{4}$ without resolving from scratch.

(c) Find the new optimal solution after adding the constraint $x_1 + x_2 = 3$.

(d) Suppose that a new variable x_4 is introduced with cost coefficient $c_4 = 4$ and column $A_4 = [1 \quad 4]^\top$ in the constraint matrix. Find the new optimal solution.

2. The following tableau corresponds to an **optimal** basis for a linear programming problem, where x_1, x_2, and x_3 are the original primal variables, and s_1 and s_2 are the slack variables corresponding to the two constraints.

	x_1	x_2	x_3	s_1	s_2
$\frac{65}{2}$	0	2	0	3	2
$\frac{5}{2}$	0	$\frac{1}{4}$	1	$\frac{1}{2}$	0
$\frac{5}{2}$	1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$\frac{1}{3}$

(a) Write the original linear program.

(b) Write the dual of the original program.

(c) Explain why the reduced costs of the slack variables in the tableau are equal to the negative of values of the dual variables for the corresponding constraints.

(d) From the given tableau, determine B^{-1}, where B is the given optimal basis.

(e) Show directly that this basis is optimal by showing how to compute the corresponding primal and dual solutions and then applying the complementary slackness conditions.

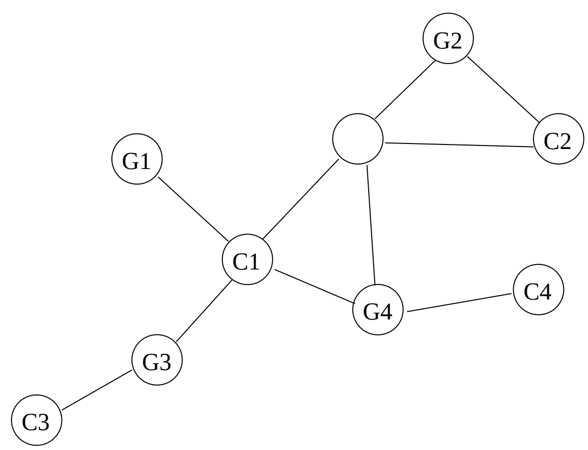
(f) Consider adding a new variable x_4 to the original problem with corresponding column $[1\ \ 1]^T$ in the constraint matrix and (unknown) cost coefficient c_4. Calculate the (expanded) tableau corresponding to B, the given basis, by calculating the entries for the new column corresponding to x_4 (the reduced cost in row zero must be expressed as a function of c_4, the unknown cost coefficient).

(g) Determine for what values of c_4 the current basis B remains optimal.

3. The following represents an electrical power distribution network connecting power generating points G_1 to G_4 with power consuming points C_1 through C_4. The arcs are undirected, i.e., power may flow in either direction. The generating capacities and unit costs are given by the following table.

	Generator			
	G_1	G_2	G_3	G_4
Capacity (thousands of KWH)	100	60	80	150
Unit Cost ($ per thousand KWH)	15.0	13.5	21.0	23.5

The power consumption for points C_1 through C_4 is 35K, 50K, 60K, and 40K respectively. There is no capacity limitation on the transmission lines and the unit cost of transmission is $11 per 1000 KWH on all lines. Note that supply exceeds demand.



(a) (10 points) Formulate the problem of finding the optimal power allocation as a minimum cost network flow problem (Hint: you may have to introduce additional arcs and/or nodes).

(b) (10 points) Find the optimal power allocation. You may use any method at all to find the optimal solution, as long as you prove rigorously that your solution is optimal. Comment on any special structure that makes this problem easier than the general case.

4. The following questions are to be answered TRUE or FALSE. You will be graded primarily on your justification, not your answer.

   (a) An iteration of the simplex algorithm can change the solution itself, i.e., have step-size $\theta^* > 0$, while leaving the cost unchanged.

   (b) If $\hat{x} \in \mathbb{R}^n$ is an optimal solution to an LP over a polyhedron $\mathcal{P}$ and $\hat{x}$ has more than m components positive, then $\hat{x}$ is the unique optimum.

   (c) If an LP over a polyhedron $\mathcal{P}$ is unbounded, it is possible to change the right-hand side to make the solution finite.

   (d) Suppose x_1, x_2, and x_3 are primal variables and s_1 and s_2 are slack variables corresponding to two inequality constraints. Then there is a linear program for which the following tableau represents a basic feasible solution.

	x_1	x_2	x_3	s_1	s_2
15	0	-1	0	3	4
3	0	$\frac{1}{4}$	1	$\frac{1}{2}$	$-\frac{1}{4}$
5	1	$-\frac{1}{2}$	0	$-\frac{1}{3}$	$\frac{1}{6}$

   (e) The problem of minimizing $\max\{c^T x, d^T x\}$ over a polyhedron $\mathcal{P}$ for given vectors $c, d \in \mathbb{R}^n$ can be written as a linear program.

   (f) If the problem of minimizing $\max\{c^T x, d^T x\}$ over a bounded polyhedron $\mathcal{P}$ for given vectors $c, d \in \mathbb{R}^n$ has an optimal solution, then it has an optimal solution that is an extreme point of $\mathcal{P}$.

   (g) In the primal simplex algorithm, a variable that has just left the basis cannot reenter in the very next iteration.

   (h) If the variables in an LP all have finite upper and lower bounds, then the LP is infeasible if and only if its dual is infeasible.

   (i) If a basic solution to a feasible (minimization) LP has an objective function value less than the optimal value, then the associated complementary dual basic solution is feasible.

   (j) If the LP $\min\{c^T x : Ax \geq b\}$ is feasible and c is a multiple of one of the rows of A, then the LP has multiple optimal solutions.

   (k) Removing a constraint that is binding at optimality from an LP results in a change in the optimal solution value.

5. This problem demonstrates that the dual simplex algorithm is not exactly the same as the simplex algorithm applied directly to the dual. Consider the following standard

form problem

$$\begin{aligned} min \quad & x_1 + x_2 \\ s.t. \quad & x_1 = 1 \\ & x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) Show that the primal problem has only one basis and that therefore, the dual simplex algorithm terminates immediately.

(b) Take the dual and show that when it is converted to standard form and solved using primal simplex, multiple pivots are required.

6. Consider the following LP.

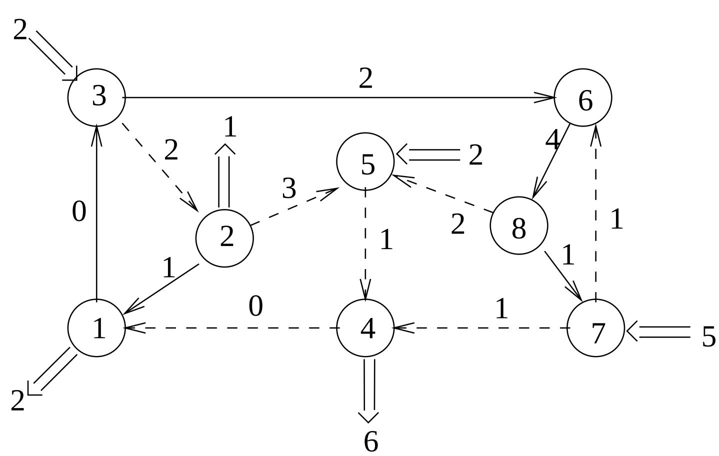
$$\begin{aligned} min \quad & 5x_1 + 4x_2 \\ s.t. \quad & x_1 + x_2 - x_3 = 1 \\ & x_1 - x_2 - x_4 = 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) Show how to avoid using the big-M method or the two phase primal simplex method by setting up an initial *dual feasible* tableau.

(b) Perform one pivot of the dual simplex algorithm.

(c) Explain why this only works for problems with nonnegative cost functions.

7. Consider the minimum cost network flow problem shown below. In this figure, the numbers next to the arcs are the costs and the numbers next to the arrows are the supplies/demands, as on the homework. All arc flows have zero lower bound and infinite upper bound.



(a) Solve the problem using the network simplex algorithm. Start with the tree indicated by the dashed arcs in the figure.

(b) Suppose the supply at node 3 is changed to 3 and the demand at 4 is changed to 7. Explain how to find the new optimal solution using only a shortest path algorithm.

(c) For each nonbasic arc, determine the range of costs for which the current basis remains optimal.

8. Consider the linear program $\max_{x \in \mathcal{P}} c^T x$ where $\mathcal{P}$ is a polyhedron with extreme points x_1, x_2, x_3, and x_4, and extreme directions d_1, d_2, and d_3 such that

$$
\begin{aligned}
c^T x_1 &= 5, && (1)\\
c^T x_2 &= 7, && (2)\\
c^T x_3 &= 4, && (3)\\
c^T x_4 &= 7, && (4)\\
c^T d_1 &= 0, && (5)\\
c^T d_2 &= -3, \text{ and} && (6)\\
c^T d_3 &= 0. && (7)
\end{aligned}
$$

(a) Characterize the set of all alternative optimal solutions to this problem.

(b) Explain why the optimal dual solution must be degenerate.

9. Consider the linear program $\min\{c^T x : Ax \geq b\}$ and a feasible solution $\hat{x}$ to this linear program. Let I be the set of indices of the binding constraints at $\hat{x}$, i.e., $I = \{i : a_i \hat{x} = b_i\}$. We will say that $\hat{x}$ is an *interior solution* if $|I| = 0$ and a *boundary solution* otherwise.

(a) Show that c is nonzero if and only if no interior solution is optimal.

(b) Show that if x^* is an optimal boundary solution such that $|I| < n$, where n is the number of variables, then there exists an infinite number of optimal solutions.

10. (35 points) This problem examines the use of the dual simplex algorithm to solve a minimization LP. The dual simplex algorithm can be viewed as a *dual ascent* method because it modifies the dual solution in a prescribed manner at each step in order to achieve an increase in the dual objective function value until an optimal solution is reached. Using an improving search paradigm, the dual solution is modified by determining an improving feasible direction (for the dual) and moving in that direction as far as possible.

Consider the following linear program and the sequence of tableaux obtained when solving it using the dual simplex algorithm. Note that the rows are presented as vectors of integers along with a scale factor by which that row is multiplied in the tableau. This is done to make the numbers easier to read. PLEASE DON'T FORGET ABOUT THE SCALE FACTOR.

$$\min\ 3x_1 + 4x_2 + 6x_3 + 7x_4 + x_5$$
$$2x_1 - x_2 + x_3 + 6x_4 - 5x_5 \geq 6$$
$$x_1 + x_2 + 2x_3 + x_4 + 2x_5 \geq 3$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Initial tableau:

Scale factor	x_1	x_2	x_3	x_4	x_5	s_1	s_2	RHS
1	3	4	6	7	1	0	0	0
1	-2	1	-1	-6	5	1	0	-6
1	-1	-1	-2	-1	-2	0	1	-3

Tableau after first pivot:

Scale factor	x_1	x_2	x_3	x_4	x_5	s_1	s_2	RHS
$\frac{1}{6}$	4	31	29	0	41	7	0	-42
$\frac{1}{6}$	2	-1	1	6	-5	-1	0	6
$\frac{1}{6}$	-4	-7	-11	0	-17	-1	6	-12

Tableau after second pivot (optimal):

Scale factor	x_1	x_2	x_3	x_4	x_5	s_1	s_2	RHS
1	0	4	3	0	4	1	1	-9
$\frac{1}{4}$	0	-3	-3	4	-9	-1	2	0
$\frac{1}{4}$	4	7	11	0	17	1	-6	12

(a) Determine the sequence of dual solutions and the improving direction for the dual solution followed in each iteration.

(b) Explain how to determine the improving direction for the dual solution by examining the tableau.

(c) Graphically depict the dual polyhedron and show the path followed by the dual simplex algorithm.

(d) What is the minimum number of pivots the dual simplex algorithm must make in order to achieve optimality? What is your conclusion about the method used for selecting the pivot row?

(e) Is the optimal *primal solution* shown in the final tableau unique? How can you tell by looking at the final tableau?

(f) Is the optimal *basis* shown in the final tableau unique? How can you tell by looking at the final tableau?

(g) Are any of the dual constraints redundant? How would you interpret this in terms of the primal problem?

11. We now consider using the primal-dual algorithm to solve the LP from problem 10. The primal-dual algorithm can also be viewed as a dual ascent algorithm that uses an improving search paradigm. However, it determines the improving direction for the dual solution directly by solving an auxiliary LP.

    (a) Taking the initial dual solution to be $(0, 0)$, convert the problem to standard form and construct the initial restricted primal problem and take its dual.

    (b) Find the optimal solution to the dual of the restricted primal in order to determine the improving direction for the dual solution (this can be done by inspection).

    (c) Determine the step size for the direction computed in the last step and compute the new dual solution, i.e., determine how far in that direction it is possible to go before violating dual feasibility and then add the proper multiple of the search direction to the previous solution (which is currently $(0, 0)$).

    (d) Form the new restricted primal problem and show that it has a feasible solution. Explain why this solution must be optimal.

    (e) Graphically depict the path in the dual polyhedron taken by the primal-dual algorithm and compare it to the path taken by the dual simplex algorithm.

    (f) By examining the dual polyhedron, can you say what would happen in both the dual simplex algorithm and the primal-dual algorithm if the objective function value of variable x_4 were changed to 6?

    (g) Does the dual solution determined at each step of the primal-dual algorithm have to be basic?

12. For this problem, consider again the linear program from problem 10.

    (a) Determine the range of objective function coefficients for x_1 for which the current basis remains optimal.

    (b) Formulate a linear program that will determine the largest total perturbation that can be made to the right hand side such that the optimal basis depicted in the final tableau remains feasible, i.e., formulate an LP that determines δ_1 and δ_2 such that the optimal basis remains feasible when the right hand side is changed to $(6 + \delta_1, 3 + \delta_2)^\top$ and such that $\delta_1 + \delta_2$ is maximized.

13. The following questions are to be answered TRUE or FALSE. Please indicate your answer clearly and provide justification for it.

    (a) For *any* given LP, if $\hat{x}$ is a feasible solution that is *not* optimal, then there exists an improving, feasible direction at $\hat{x}$.

    (b) For a given LP in *standard form* and a given basis matrix B, if both the primal and dual solutions corresponding to B are feasible, then they are both optimal.

(c) For a given LP in *standard form* and a given basis matrix B, if the dual solution corresponding to B is infeasible, then the primal solution cannot be optimal.

(d) For *any* given LP, if there exists an optimal solution, there exists an optimal solution that is basic feasible.

14. Consider the following simple linear program:

$$
\begin{aligned}
\min\ & -2x_1 - x_2 \\
s.t.\quad & x_1 + x_2 \leq 2, \\
& x_1 - x_2 \leq 0, \\
& x_1 \leq 1, \text{ and} \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

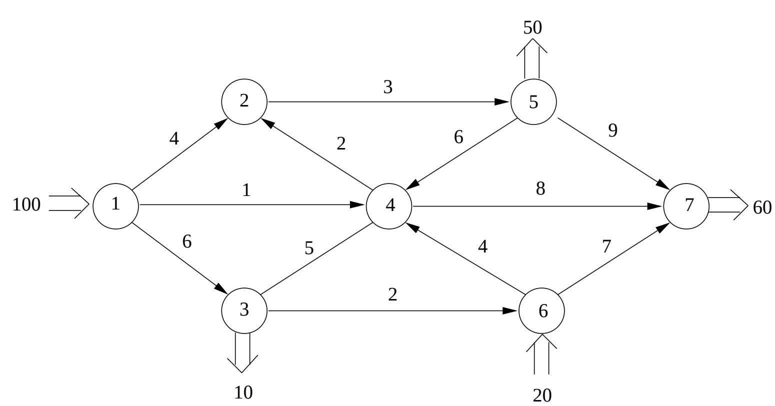
(a) Write the dual of this linear program.

(b) By inspection, determine all basic feasible solutions to the dual. Which of these solutions are optimal? (Be sure your stated solutions are solutions to the dual of the LP in the form given. Changing the LP to an equivalent form may change the sign of the dual solution.) EXPLAIN YOUR METHOD.

(c) Using complementarity arguments, determine the optimal primal solution. SHOW YOUR WORK.

(d) Starting from the dual solution $(-2, 0, 0)$, use the primal-dual algorithm to solve this LP. Note that the LPs you need to solve to do this are small and can be solved by inspection. You do not have to solve them formally. SHOW YOUR WORK.

15. Consider the following representation of a minimum cost network flow problem.



The numbers inside each circle represent the node numbers, the numbers next to each arc represent the cost of a unit of flow on that arc, and the remaining numbers represent supply/demand at each node (those without numbers are transshipment nodes).

(a) An optimal dual solution to this problem is

$$\{0, -3, -6, -1, -6, -2, -9\}.$$

Is this dual solution degenerate? Why or why not?

(b) Using complementarity, determine an optimal primal solution. Are there alternative optima? JUSTIFY YOUR ANSWER.

(c) How much can the cost of arc $(1, 2)$ be decreased while still maintaining the optimality of this solution? SHOW YOUR WORK.

(d) How would the solution change if the cost of arc $(1, 2)$ were changed to 2? SHOW YOUR WORK.

16. Consider the following tableau associated with a given LP.

x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	0	0	1	-1	1	3
0	1	0	2	2	-1	4
0	0	1	-1	1	2	6

(a) What is the basic feasible solution displayed in this tableau?

(b) Assuming x_4, x_5, and x_6 are slack variables corresponding to "$\leq$" constraints in the original LP, what is the dual solution corresponding to the current basis if the objective function is

$$3x_1 - 2x_2 - 10x_3 - 11x_4 + 18x_5 - 10x_6$$

SHOW YOUR WORK.

(c) Find all the extreme points of the feasible region that are adjacent to the extreme point indicated by the above tableau. SHOW YOUR WORK.

(d) If the original right-hand side is b, what is the maximum amount by which b_1 can be changed (in each direction) such that the current basis remains feasible? SHOW YOUR WORK.

17. Consider the integer program

$$\begin{aligned} \max \quad & x_2 \\ \text{s.t.} \quad x_1 + x_2 & \leq 7/2, && (8) \\ -x_1 + x_2 & \leq 3/2, && (9) \\ x_1 + x_2 & \geq 1, && (10) \\ x_1, x_2 & \in \mathbb{Z}_+. && (11) \end{aligned}$$

(a) Describe the convex hull of solutions to this integer program.

(b) What is the optimal solution to the LP relaxation of this integer program? What is the optimal solution to the integer program itself? JUSTIFY YOUR ANSWER.

(c) If we decrease the right-hand side of constraint (8) by δ, how much does the solution to the LP relaxation change, assuming the optimal basis remains the same? SHOW YOUR WORK.

(d) How much does δ have to decrease before the solution to the integer program changes? JUSTIFY YOUR ANSWER.