Introduction to Mathematical Programming
IE496

Lecture 6

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Reading for This Lecture

- Bertsimas 3.1-3.2.

What We've Learned So Far

- We are interested in the extreme points of polyhedra.
- There is a one-to-one correspondence between the extreme points of a polyhedron and the basic feasible solutions.
- We can construct basic solutions by
  - Choosing a basis B of m linearly independent columns of A.
  - Solve the system $Bx_B = b$ to obtain the values of the basic variables.
  - Set $x_N = 0$.
- We can move between adjacent (nondegenerate) basic solutions by removing one column of the basis and replacing it with another.
- In the presence of degeneracy, we might stay at the same extreme point.
- These are the building blocks we need to construct algorithms for solving LPs.

Iterative Search Algorithms

- Many optimization algorithms are *iterative* in nature.
- Geometrically, this means that they move from a given starting point to a new point in a specified *search direction*.
- This search direction is calculated to be both *feasible* and *improving*.
- The process stops when we can no longer find a feasible, improving direction.
- For linear programs, it is always possible to find a feasible improving direction if we are not at an optimal point.
- This is essentially what makes linear programs "easy" to solve.

Feasible and Improving Directions

Definition 1. *Let $\hat{x}$ be an element of a polyhedron $\mathcal{P}$. A vector $d \in \mathbb{R}^n$ is said to be a* feasible direction *if there exists $\theta \in \mathbb{R}_+$ such that $\hat{x} + \theta d \in \mathcal{P}$.*

Definition 2. *Consider a polyhedron $\mathcal{P}$ and the associated linear program $\min_{x \in \mathcal{P}} c^\top x$ for $c \in \mathbb{R}^n$. A vector $d \in \mathbb{R}^n$ is said to be an* improving direction *if $c^\top d < 0$.*

Notes:

Constructing Feasible Search Directions

- Consider a BFS $\hat{x}$, so that $\hat{x}_N = 0$.
- Any feasible direction must increase the value of at least one of the nonbasic variables (why?).
- We will consider moving in *basic directions* that increase the value of exactly one of the nonbasic variables, say variable j. This means
- In order to remain feasible, we must also have $Ad = 0$ (why?), which means

Constructing Improving Search Directions

- Now we know how to construct feasible search directions—how do we ensure they are improving?
- Recall that we must have $c^\top d < 0$.

Definition 3. *Let $\hat{x}$ be a basic solution, let B be an associated basis matrix, and let c_B be the vector of costs of the basic variables. For each j, we define the* reduced cost *$\bar{c}_j$ of variable j by*

$$\bar{c}_j = c_j - c_B^\top B^{-1} A_j.$$

- The basic direction associated with variable j is improving if and only if $\bar{c}_j < 0$.
- Note that all basic variables have a reduced cost of 0 (why?).

Optimality Conditions

Theorem 1. *Consider a basic feasible solution $\hat{x}$ associated with a basis matrix B and let $\bar{c}$ be the corresponding vector of reduced costs.*

- *If $\bar{c} \geq 0$, then $\hat{x}$ is optimal.*
- *If $\hat{x}$ is optimal and nondegenerate, then $\bar{c} \geq 0$.*

Notes:

Optimal Bases

Definition 4. *A basis matrix* B *is said to optimal is*

- $B^{-1}b \geq 0$, *and*
- $\bar{c} \geq 0$.

Notes:

An Algorithm for Linear Programming

We will develop the following basic algorithm for linear programming:

1. Find an initial BFS.
2. Compute the reduced costs.
3. Determine an improving feasible direction d.
4. Move as far as possible in direction d to a new BFS.
5. If the new BFS is not optimal, then repeat.

The Step Length

- For now, we will assume that we can find an initial BFS (Step 1).
- We will also assume nondegeneracy.
- We have already seen how to compute the reduced costs and find an improving feasible direction (Steps 2 and 3).
- The distance we move in the computed direction is the *step length*.
- We want to move as far as possible, so the step length is
- What determines the step length?

Determining the Step Length

- If $d \geq 0$, then the step length is ∞ and the linear program is unbounded.
- if $d_i < 0$, then $\hat{x}_i + \theta d_i \geq 0 \Rightarrow \theta \leq -\frac{\hat{x}_i}{d_i}$.
- Therefore, we can compute the step length explicitly as
- Note that we need only consider the basic variables in this computation.

Determining the Next Solution

- Once we have θ^*, the new feasible solution is $\hat{x} + \theta^* d$. Is this a BFS?
- One variable that was nonbasic now has positive value (the *entering variable*.
- One (at least) variable that was basic now has value 0 (the *leaving variable*).
- If j is the entering variable and i is the leaving variable, define the new set of basic variables by
- Is the corresponding matrix $\bar{B}$ a basis matrix?

Determining the Next Basis

Theorem 2.

- *The columns* $A_{\bar{B}(i)}$ *for* $i \in [1..m]$ *are linearly independent and hence* $\bar{B}$ *does form a basis matrix.*
- *The vector* $\hat{x} + \theta^* d$ *is the BFS corresponding to* $\bar{B}$*.*

Notes:

The Simplex Method

A typical iteration of the simplex method:

1. Start with a specified basis matrix B and a corresponding BFS x^0.
2. Compute the reduced cost vector $\bar{c}$. If $\bar{c} \geq 0$, then x^0 is optimal.
3. Otherwise, choose j for which $\bar{c}_j < 0$.
4. Compute $u = B^{-1}A_j$. If $u \leq 0$, then $\theta^* = \infty$ and the LP is unbounded.
5. Otherwise, $\theta^* = \min_{\{i=1,\ldots,m:u_i>0\}} \frac{x^0_{B(i)}}{u_i}$.
6. Choose l such that $\theta^* = \frac{x^0_B(l)}{u_l}$ and form a new basis, replacing $A_{B(l)}$ with A_j. The values of the new basic variables are $x^1_j = \theta^*$ and $x^1_{B(i)} = x^0_{B(i)} - \theta^* u_i$ if $i \neq l$.