# Introduction to Mathematical Programming IE406

## Lecture 4

Dr. Ted Ralphs

# Reading for This Lecture

- Bertsimas 2.2-2.4

# The Two Crude Petroleum Example Revisited

- Recall the Two Crude Petroleum example.
- We showed graphically that the optimal solution was an extreme point.
- How did we figure out the coordinates of the optimal point?

# Binding Constraints

Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq b\}$.

**Definition 1.** *If a vector* $\hat{x}$ *satisfies* $a_i^\top \hat{x} = b_i$*, then we say the corresponding constraint is* binding *at* $\hat{x}$*.*

**Theorem 1.** *Let* $\hat{x} \in \mathbb{R}^n$ *be given and let* $I = \{i \mid a_i^\top \hat{x} = b_i\}$ *represent the set of constraints that are binding at* $\hat{x}$*. Then the following are equivalent:*

- *There exist* $n$ *vectors in the set* $\{a_i \mid i \in I\}$ *that are linearly independent.*
- *The span of the vectors* $\{a_i \mid i \in I\}$ *is* $\mathbb{R}^n$*.*
- *The system of equations* $a_i^\top x = b_i, i \in I, x \in \mathbb{R}^n$ *has the unique solution* $\hat{x}$*.*

If the vectors $\{a_j \mid j \in J\}$ for some $J \subseteq [1, m]$ are linearly independent, we will say that the corresponding constraints are also linearly independent.

# Basic Solutions

Consider a polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq b\}$ and let $\hat{x} \in \mathbb{R}^n$ be given.

**Definition 2.** *The vector $\hat{x}$ is a* basic solution *with respect to $\mathcal{P}$ if there exist $n$ linearly independent, binding constraints at $\hat{x}$.*

**Definition 3.** *If $\hat{x}$ is a basic solution and $\hat{x} \in \mathcal{P}$, then $\hat{x}$ is a* basic feasible solution*.*

**Theorem 2.** *If $\mathcal{P}$ is nonempty and $\hat{x} \in \mathcal{P}$, then the following are equivalent:*

- *$\hat{x}$ is a vertex.*
- *$\hat{x}$ is an extreme point.*
- *$\hat{x}$ is a basic feasible solution.*

# Adjacent Basic Solutions

- Two distinct basic solutions $x$ and $y$ are *adjacent* if there are $n-1$ linearly independent constraints that are binding at both $x$ and $y$.
- If two adjacent basic solutions are also feasible, then the line connecting them is called an *edge* of the polyhedron.
- Note that the first algorithms we will study move through the polyhedron along its edges.

# Some Observations

Note the immediate consequences of the previous results:

# Polyhedra in Standard Form

- For the next few slides, we consider the standard form polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$.
- Recall that the feasible region of any linear program can be expressed equivalently in this form.
- We will assume that the rows of $A$ are linearly independent $\Rightarrow m \leq n$.
- Later, we will show that any polyhedron in standard form can be represented in this way.
- What does a basic feasible solution look like here?

# Basic Feasible Solutions in Standard Form

- In standard form, the equations are always binding.
- To obtain a basic solution, we must set $n - m$ of the variables to zero (why?).
- We must also end up with a set of linearly independent constraints.
- Therefore, the variables we pick cannot be arbitrary.

**Theorem 3.** *Consider a polyhedron $\mathcal{P}$ in standard form with $m$ linearly independent constraints. A vector $\hat{x} \in \mathbb{R}^n$ is a basic solution with respect to $\mathcal{P}$ if and only if $A\hat{x} = b$ and there exist indices $B(1), \ldots, B(m)$ such that:*

- *The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent, and*
- *If $i \neq B(1), \ldots, B(m)$, then $\hat{x}_i = 0$.*

# Basic Feasible Solutions in Standard Form

- As a consequence of the previous theorem, we now know how to construct basic solutions for polyhedra in standard form.
  -
  -
  -
- If the resulting solution is also nonnegative, then it is a basic feasible solution.

# Some Terminology

- If $\hat{x}$ is a basic solution, then $\hat{x}_{B(1)}, \ldots, \hat{x}_{B(m)}$ are the *basic variables*.
- The columns $A_{B(1)}, \ldots, A_{B(m)}$ are called the *basic columns*.
- Since they are linearly independent, these columns form a *basis* for $\mathbb{R}^m$.
- A set of basic columns form a *basis matrix*, denoted $B$. So we have,

$$B = \left[A_{B(1)}\ A_{B(2)} \cdots A_{B(m)}\right], \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}$$

# Basic Solutions and Bases

- Given a basis matrix $B$, the values of the basic variables are obtained by solving $Bx_B = b$, whose unique solution is $x_B = B^{-1}b$.
- However, multiple bases can give the same basic solution.
- Two bases are *adjacent* if they differ in only one basic column.
- Two basic solutions are adjacent if and only if they can be obtained from two adjacent bases (proof is homework).

# The Full Row Rank Assumption

**Theorem 4.** *Let* $\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ *for some* $A \in \mathbb{R}^{m \times n}$ *with* $rank(A) = k$. *If rows* $a_{i_1}^\top, a_{i_2}^\top, \ldots, a_{i_k}^\top$ *are linearly independent, then*

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid a_{i_1}^\top x = b_{i_1}, a_{i_2}^\top x = b_{i_2}, \ldots, a_{i_k}^\top x = b_{i_k}, x \geq 0\}.$$

Notes:

# Degeneracy

**Definition 4.** *A basic solution $\hat{x}$ is called* degenerate *if more than $n$ of the constraints are binding at $\hat{x}$.*

Notes: