Introduction to Mathematical Programming IE406

Lecture 3

Dr. Ted Ralphs

Reading for This Lecture

- Bertsimas 2.1-2.2

From Last Time

- Recall the Two Crude Petroleum example.
- In the example, the optimal solution was a "corner point."
- We saw that the following are possible outcomes of solving an optimization problem:
  -
  -
  -
  -
- In fact, we will see that these are the only possibilities.
- We will also see that when there is an optimal solution and at least one "corner point," there is an optimal solution that is a "corner point."

Some Definitions

Definition 1. *A* polyhedron *is a set of the form* $\{x \in \mathbb{R}^n | Ax \geq b\}$, *where* $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$.

Definition 2. *A set* $S \subset \mathbb{R}^n$ *is* bounded *if there exists a constant* K *such that* $|x_i| < K \ \forall x \in S, \forall i \in [1, n]$.

Definition 3. *Let* $a \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$ *be given.*

- *The set* $\{x \in \mathbb{R}^n | a^\top x = b\}$ *is called a* hyperplane.
- *The set* $\{x \in \mathbb{R}^n | a^\top x \geq b\}$ *is called a* half-space.

Notes:

Convex Sets

Definition 4. *A set* $S \subseteq \mathbb{R}^n$ *is* convex *if* $\forall x, y \in S$ *and* $\lambda \in \mathbb{R}$ *with* $0 \leq \lambda \leq 1$*, we have* $\lambda x + (1 - \lambda)y \in S$*.*

Definition 5. *Let* $x^1, \ldots, x^k \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}^k$ *be given such that* $\lambda^\top \mathbf{1} = 1$*.*

- *The vector* $\sum_{i=1}^k \lambda_i x^i$ *is said to be a* convex combination *of* $x^1, \ldots, x^k$*.*
- *The* convex hull *of* $x^1, \ldots, x^k$ *is the set of all convex combinations of these vectors.*

Notes:

Properties of Convex Sets

The following properties can be derived from the definitions:

- The intersection of convex sets is convex.
- Every polyhedron is a convex set.
- The convex combination of a finite number of elements of a convex set also belongs to the set.
- The convex hull of a finite number of vectors is a convex set.

How do we prove each of these?

Aside: Mathematical Proofs

- A mathematical proof shows the correctness of a given statement based on known definitions, axioms, and previously proven statements.
- Most proofs are for statements of the form $A \Rightarrow B$ where A and B are both statements.
- Example: "If $x > 2$ is a real number, then there exists a real number $y < 0$ such that $x = \frac{2y}{1+y}$".
- Proof:
- What are A and B in this example?

Mathematical Proofs: Quantifying Variables

- *Quantifying* is specifying from which set and for which values of a variable a statement is true.
- Example: "For all real numbers x and y, $(x+y)^2 = x^2 + 2xy + y^2$."
- This specifies that x and y can have any real value.
- Example: "For all real numbers $x \geq 0$, $x = |x|$."
- This specifies that the statement is true for nonnegative values of x.

Mathematical Proofs: Types of Quantifiers

- Universal Quantifiers
  - Statements that include "for all" or "for every."
  -
  - Example: "For all real numbers x, $cos^2 x + sin^2 x = 1$."
- Existential Quantifiers
  - Statements that include "there exists" or "there is."
  -
  - Example: "For every real number $0 \leq x \leq 1$, there exists a real number $0 \leq y \leq \frac{\pi}{2}$ such that $sin(y) = x$."
- Notation: $\forall$ means "for all" and $\exists$ means "there exists".
- Example: "$\forall x \in \mathbb{R}$ such that $0 \leq x \leq 1$, $\exists y \in \mathbb{R}$ such that $0 \leq y \leq \frac{\pi}{2}$ and $sin(y) = x$."

Mathematical Proofs: Proofs with Universal Quantifiers

- To prove something about a universally quantified statement, first let an arbitrary set element *be given*.
- Example: "If $C \in \mathbb{R}^{n \times n}$ and $det(C) \neq 0$, then $\exists C^{-1} \in \mathbb{R}^{n \times n}$ such that $CC^{-1} = I$."
- Start of Proof: "Let an arbitrary matrix $C \in \mathbb{R}^{n \times n}$ be given such that $det(C) \neq 0$..."
- Now prove that statement is true for the given element.
- Since the element was arbitrary, this proves the original statement.

Mathematical Proofs: Proofs with Existential Quantifiers

- If you are trying to prove something about an existentially quantified variable, the proof is often *constructive*.
- The proof gives a technique for constructing an element of the set with the given property.
- Example: "If $C \in \mathbb{R}^{n \times n}$ and $det(C) \neq 0$, then $\exists C^{-1} \in \mathbb{R}^{n \times n}$ such that $CC^{-1} = I$."
- Proof Technique: Construct C^{-1}.

Mathematical Proofs: Choosing an Element

- If you know from a previous theorem that an element of a set with a particular property exists, then you may "*choose*" it.
- Example: "Let r, a positive rational number be given. Then we may choose natural numbers p and q such that $r = \frac{p}{q}$."
- This can be especially useful in constructive proofs.

Mathematical Proofs: Proof Techniques

- Prove the contrapositive.
- Proof by contradiction.
- Proof by induction.
- Proof by cases.
- Other types of proofs
  - Uniqueness proofs.
  - Either/or proofs.
  - If and only if proofs.

Back to Our Story

Let's prove the following:

Proposition 1. *The intersection of convex sets is convex.*

Proof:

Proposition 2. *Every polyhedron is convex.*

Proof:

Extreme Points and Vertices

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a given polyhedron.

Definition 6. *A vector $x \in \mathcal{P}$ is an* extreme point *of $\mathcal{P}$ if $\nexists y, z \in \mathcal{P}, \lambda \in (0, 1)$ such that $x = \lambda y + (1 - \lambda)z$.*

Definition 7. *A vector $x \in \mathcal{P}$ is a* vertex *of $\mathcal{P}$ if $\exists c \in \mathbb{R}^n$ such that $c^\top x < c^\top y \; \forall y \in \mathcal{P}, x \neq y$.*

Notes:

A Little Linear Algebra Review

Definition 8. *A finite collection of vectors* $x_1, \ldots, x_k \in \mathbb{R}^n$ *is* linearly independent *if the unique solution to* $\sum_{i=1}^{k} \lambda_i x^i = 0$ *is* $\lambda_i = 0, i = 1, \ldots, k$*. Otherwise, the vectors are* linearly dependent*.*

Let A be a square matrix. Then, the following statements are equivalent:

- The matrix A is invertible.
- The matrix $A^\top$ is invertible.
- The determinant of A is nonzero.
- The rows of A are linearly independent.
- The columns of A are linearly independent.
- For every vector b, the system $Ax = b$ has a unique solution.
- There exists some vector b for which the system $Ax = b$ has a unique solution.

A Little More Linear Algebra Review

Definition 9. *A nonempty subset* $S \subseteq \mathbb{R}^n$ *is called a* subspace *if* $\alpha x + \gamma y \in S \; \forall x, y \in S$ *and* $\forall \alpha, \gamma \in \mathbb{R}$.

Definition 10. *A* linear combination *of a collection of vectors* $x^1, \ldots, x^k \in \mathbb{R}^n$ *is any vector* $y \in \mathbb{R}^n$ *such that* $y = \sum_{i=1}^{k} \lambda_i x^i$ *for some* $\lambda \in \mathbb{R}^k$.

Definition 11. *The* span *of a collection of vectors* $x^1, \ldots, x^k \in \mathbb{R}^n$ *is the set of all linear combinations of those vectors.*

Definition 12. *Given a subspace* $S \subseteq \mathbb{R}^n$, *a collection of linearly independent vectors whose span is* S *is called a* basis *of* S. *The number of vectors in the basis is the* dimension *of the subspace.*

Subspaces and Bases

- A given subspace has an infinite number of bases.
- Each basis has the same number of vectors in it.
- If S and T are subspaces such that $S \subset T \subset \mathbb{R}^n$, then a basis of S can be extended to a basis of T.
- The span of the columns of a matrix A is a subspace called the *column space* or the *range*, denoted $range(A)$.
- The span of the rows of a matrix A is a subspace called the *row space*.
- The dimensions of the column space and row space are always equal. We call this number $rank(A)$.
- Clearly, $rank(A) \leq \min\{m, n\}$. If $rank(A) = \min\{m, n\}$, then A is said have *full rank*.
- The set $\{x \in \mathbb{R}^n | Ax = 0\}$ is called the *null space* of A (denoted $null(A)$) and has dimension $n - rank(A)$.

Some Conventions

If not otherwise stated, the following conventions will be followed for lecture slides during the course:

- $\mathcal{P}$ will denote a polyhedron contained in $\mathbb{R}^n$.
- A will denote a matrix of dimension m by n.
- b will denote a vector of dimension m.
- x will denote a vector of dimension n.
- c will denote a vector of dimension n.
- $\mathcal{P}$ will either be defined in *standard form* ($\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$) or *inequality form* ($\{x \in \mathbb{R}^n | Ax \geq b\}$).
- We will usually be minimizing.