Introduction to Mathematical Programming
IE406

Lecture 2

Dr. Ted Ralphs

Reading for This Lecture

- Primary Reading
  - Bertsimas 1.1-1.2, 1.4-1.5
- Supplementary Reading
  - Bertsimas 1.3
  - Operations Research Methods and Models
  - Model Building in Mathematical Programming

Review from Last Time

- Recall that a mathematical model consists of:
  - Decision variables (with domains)
  - Constraints (functions of the variables with domains)
  - Objective Function (maximize or minimize)
  - Parameters and Data

The general form of a *mathematical programming model* is:

$$\begin{array}{lrcl} min \text{ or } max & f(x_1, \ldots, x_n) & & \\ s.t. & g_i(x_1, \ldots, x_n) & \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} & b_i \\ & (x_1, \ldots, x_n) & \in & X \end{array}$$

where X may be a discrete set.

Example of a Mathematical Program: The Diet Problem

- Goal: Choose the cheapest menu satisfying nutritional requirements.
- What is the input data?
- What is the formulation in words?

Critique of the Model

What are the possible problems with this model?

A Little History

- George Dantzig is considered to be the father of linear programming.
- The diet problem was one of the first applications of linear programming.
- It took *120 man-days* to solve a problem with 9 constraints and 77 variables by hand!
- Later, Dantzig tried to lose weight by designing his own diet.
  - The first solution he came up with contained several gallons of vinegar.
  - After deleting vinegar from the list of foods, the new solution contained approximately 200 bouillon cubes.
  - This illustrates one of the potential hazards of math modeling.

Representing Math Models: Vectors and Matrices

- An $m \times n$ *matrix* is an array of mn real numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A is said to have n *columns* and m *rows*.
- An n*-dimensional column vector* is a matrix with one column.
- An n*-dimensional row vector* is a matrix with one row .
- By default, a *vector* will be considered a column vector.
- The set of all n*-dimensional vectors* will be denoted $\mathbb{R}^n$.
- The set of all $m \times n$ *matrices* will be denoted $\mathbb{R}^{m \times n}$.

Matrices

- The *transpose* of a matrix A is

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- If $x, y \in \mathbb{R}^n$, then $x^\top y = \sum_{i=1}^n x_i y_i$.
- This is called the *inner product* of x and y.
- If $A \in \mathbb{R}^{m \times n}$, then A_j is the j^{th} column, and $a_j^\top$ is the j^{th} row.
- If $A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times n}$, then $[AB]_{ij} = a_i^\top B_j$.

Linear Functions

- A *linear function* $f : \mathbb{R}^n \to \mathbb{R}$ is a weighted sum, written as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i$$

  for given coefficients $c_1, \ldots, c_n$.

- We can write $x_1, \ldots, x_n$ and $c_1, \ldots, c_n$ as vectors $x, c \in \mathbb{R}^n$ to obtain:

$$f(x) = c^\top x$$

- In this way, a linear function can be represented simply as a vector.
- We will consider only models defined by linear functions.

Back to the Diet Problem

- How do we write the diet problem mathematically?
- What are the decision variables?
- What is the objective function?
- What are the constraints?

Putting It All Together

- Using matrix notation, we can write our current formulation as

$$\begin{aligned} min \quad & c^\top x \\ s.t. \quad & l \leq Ax \leq u \end{aligned}$$

- What are we missing?

Linear Programs

- What we have just seen is an example of a *linear program*.
- In general, we can write a linear program as

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \geq b_i \; \forall i \in M_1 \\ & a_i^\top x \leq b_i \; \forall i \in M_2 \\ & a_i^\top x = b_i \; \forall i \in M_3 \\ & x_j \geq 0 \;\; \forall j \in N_1 \\ & x_j \leq 0 \;\; \forall j \in N_2 \end{aligned}$$

- This in turn can be written equivalently as

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

- How do we do this?

Standard Form

- To solve a linear program, it is convenient to put it in the following *standard form*:

$$\begin{aligned} min\, c^\top x & \\ s.t. \quad Ax &= b \\ x &\geq 0 \end{aligned}$$

- How do we do this?

Two Crude Petroleum Example

- Two Crude Petroleum distills crude from two sources:
  - Saudi Arabia
  - Venezuela
- They have three main products:
  - Gasoline
  - Jet fuel
  - Lubricants
- Yields

	Gasoline	Jet fuel	Lubricants
Saudi Arabia	0.3 barrels	0.4 barrels	0.2 barrels
Venezuela	0.4 barrels	0.2 barrels	0.3 barrels

Two Crude Petroleum Example (cont.)

- Availability and cost

	Availability	Cost
Saudi Arabia	9000 barrels	\$20/barrel
Venezuela	6000 barrels	\$15/barrel

- Production Requirements (per day)

Gasoline	Jet fuel	Lubricants
2000 barrels	1500 barrels	500 barrels

- Objective: Minimize production cost.

Modeling the Two Crude Production Problem

- What are the decision variables?
- What is the objective function?
- What are the constraints?

Linear Programming Formulation of Two Crude Example

- This yields the following LP formulation:

$$
\begin{aligned}
\min\ & 20000x_1 + 15000x_2 \\
\text{s.t.}\ & 0.3x_1 + 0.4x_2 \geq 2.0 \\
& 0.4x_1 + 0.2x_2 \geq 1.5 \\
& 0.2x_1 + 0.3x_2 \geq 0.5 \\
& 0 \leq x_1 \leq 9 \\
& 0 \leq x_2 \leq 6
\end{aligned}
$$

- How can we solve this problem?
- What are the possible outcomes of solving such a problem?