Introduction to Mathematical Programming
IE406

Lecture 13

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Reading for This Lecture

- Bertsimas Chapter 5

Sensitivity Analysis

- In many real-world problems, the following can occur:
  - The input data is not very accurate.
  - We don't know all of the constraints ahead of time.
  - We don't know all of the variables ahead of time.
- Because of this, we want to analyze the dependence of the model on the input data, i.e.,
  - the matrix A,
  - the right-hand side vector b, and
  - the cost vector c.
- We would also like to know the effect of additional variables and constraints.
- This is done using *sensitivity analysis*.
- Sensitivity analysis requires nothing more than straightforward application of techniques we've already developed.

The Fundamental Idea

- Using the simplex algorithm to solve a standard form problem, we know that if B is an optimal basis, then two conditions are satisfied:
  - $B^{-1}b \geq 0$
  - $c^\top - c_B^\top B^{-1}A \geq 0$
- When the problem is changed, we can check to see how these conditions are affected.
- This is the simplest kind of analysis—we have already seen several examples.
- When using the simplex method, we always have B^{-1} available, so we can easily recompute appropriate quantities.
- Where is B^{-1} in the simplex tableau?

Adding a New Variable

- Suppose we want to consider adding a new variable to the problem, e.g., we want to consider adding a new product to our line.
- We simply compute the reduced cost of the new variable as

$$c_j - c_B^\top B^{-1} A_j$$

where A_j is the column corresponding to the new variable in the matrix.

- If the reduced costs is nonnegative, then we should not consider adding the product.
- Otherwise, it is eligible to enter the basis and we can reoptimize from the current feasible (but now non-optimal) basis.

Adding a New Inequality Constraint

- Suppose we want to introduce a new constraint of the form $a_{m+1}^\top x \geq b_{m+1}$.
- The new constraint matrix (in standard form) would look like

$$\begin{bmatrix} A & 0 \\ a_{m+1}^\top & -1 \end{bmatrix}$$

- Hence, the new basis matrix would look like

$$\bar{B} = \begin{bmatrix} B & 0 \\ a^\top & -1 \end{bmatrix}$$

- The new basis inverse would then be

$$\bar{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ a^\top B^{-1} & -1 \end{bmatrix}$$

Adding a New Inequality Constraint (cont.)

- The vector of reduced costs is

$$[c^\top \ 0] - [c_B^\top \ 0] \begin{bmatrix} B^{-1} & 0 \\ a^\top B^{-1} & -1 \end{bmatrix} \begin{bmatrix} A & 0 \\ a_{m+1}^\top & -1 \end{bmatrix} = [c^\top - c_B^\top B^{-1} A \ \ 0]$$

  and so the reduced costs remain unchanged.

- Hence, we have a dual feasible basis and we apply dual simplex.

- The tableau can be computed as

$$\bar{B}^{-1} \begin{bmatrix} A & 0 \\ a_{m+1}^\top & -1 \end{bmatrix} = \begin{bmatrix} B^{-1}A & 0 \\ a^\top B^{-1} A - a_{m+1}^\top & 1 \end{bmatrix}$$

- Note that $B^{-1}A$ is available from the original tableau.

Adding a New Equality Constraint

- Assume the new constraint is not satisfied by the current optimal solution.
- We introduce an artificial variable x_{n+1}, as in the two-phase method, and consider the LP (assuming $a_{m+1}^\top x^* > b_{m+1}$)

$$\begin{aligned} & min\ c^\top x + M x_{n+1} \\ & s.t. \qquad\qquad Ax = b \\ & \qquad a_{m+1}^\top x - x_{n+1} = b_{m+1} \\ & \qquad\qquad x \geq 0, x_{n+1} \geq 0 \end{aligned}$$

- We can obtain a primal feasible basis by making the new variable basic.
- The new tableau can be computed as before.
- If the new problem is feasible and M is large enough, then the solution will have $x_{n+1} = 0$.
- The values of the remaining variables will yield an optimal solution to the original problem with the additional constraint.

Changes to the Right-hand Side

- Suppose we change b_i to $b_i + \delta$.
- The values of the basic variables change from $B^{-1}b$ to $B^{-1}(b + \delta e^i)$, where e^i is the i^{th} unit vector.
- The feasibility condition is then

$$B^{-1}(b + \delta e^i) \geq 0$$

- If g is the i^{th} column of B^{-1}, then the feasibility condition becomes

$$x_B + \delta g \geq 0$$

- This is equivalent to

$$\max_{\{j | g_j > 0\}} \left(-\frac{x_{B(j)}}{g_j} \right) \leq \delta \leq \min_{\{j | g_j < 0\}} \left(-\frac{x_{B(j)}}{g_j} \right).$$

- If δ is outside the allowable range, we can reoptimize using dual simplex.

Changes in the Cost Vector

- Suppose we change some cost coefficient from c_j to $c_j + \delta$.
- If c_j is the cost coefficient of a nonbasic variable, then we need only recalculate its reduced cost.
- The reduced cost itself increases by δ and the current solution remains optimal as long as $\delta \geq -\bar{c}_j$.
- Otherwise, we reoptimize using the primal simplex method.
- If c_j is the cost coefficient of the l^{th} basic variable, then c_B becomes $c_B + \delta e_l$ and the new optimality conditions are

$$(c_B + \delta e_l)^\top B^{-1} A \leq c^\top$$

- This is equivalent to

$$\delta q \leq \bar{c}$$

  where q is the l^{th} row of $B^{-1}A$, which is available in the simplex tableau.

Changes in a Nonbasic Column of A

- Suppose we change some entry a_{ij} of the constraint matrix to $a_{ij} + \delta$.
- If column j is nonbasic, then B does not change and we only need to check the reduced cost of column j.
- The new reduced cost is

$$c_j - c_B^\top B^{-1}(A_j + \delta e^i)$$

- This means the current solution remains optimal if

$$\bar{c_j} - \delta p_i \geq 0$$

- Otherwise, we reoptimize with primal simplex.

Changes in a Basic Column of A

- This case is more complicated and will be left to the next homework.
- Suppose x^* and p^* are optimal primal and dual solutions.
- If the basic column A_j is changed to $A_j + \delta e^i$, then if $x^*(\delta)$ is the new solution, it can be shown that

$$c^\top x^*(\delta) = c^\top x^* - \delta x_j^* p_i^* + O(\delta^2)$$

- This is in concert with our previous economic interpretations of duality and optimality.