Introduction to Mathematical Programming
IE406

Lecture 12

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Reading for This Lecture

- Bertsimas 4.8-4.9

Polyhedral Cones

Definition 1. *A set* $C \subset \mathbb{R}^n$ *is a cone if* $\lambda x \in C$ *for all* $\lambda \geq 0$ *and all* $x \in C$.

Definition 2. *A polyhedron of the form* $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq 0\}$ *is called a* polyhedral cone.

Theorem 1. *Let* $C \subset \mathbb{R}^n$ *be the polyhedral cone defined by the matrix* A. *Then the following are equivalent:*

1. *The zero vector is an extreme point of* C.
2. *The cone* C *does not contain a line.*
3. *The rows of* A *span* $\mathbb{R}^n$.

Comments on Polyhedral Cones

- Notice that the origin is a member of every polyhedral cone.
- Furthermore, the origin is the only possible extreme point.
- A polyhedral cone that has the origin as an extreme point is called *pointed*.
- Graphically, a pointed cone looks like what we would ordinarily call a cone.

The Recession Cone

- Consider a nonempty polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq b\}$ and fix a point $y \in \mathcal{P}$.
- The *recession cone* at y is the set of all directions along which we can move indefinitely from y and still be in $\mathcal{P}$, i.e.,

$$\{d \in \mathbb{R}^n | A(y + \lambda d) \geq b \ \ \forall \lambda \geq 0\}.$$

- This set turns out to be

$$\{d \in \mathbb{R}^n | Ad \geq 0\}$$

  and is hence a polyhedral cone independent of y.
- The nonzero elements of the recession cone are called the *rays* of $\mathcal{P}$.
- For a polyhedron in standard form, the rays must satisfy $Ad = 0$, $d \geq 0$.

Extreme Rays

Definition 3.

1. *A nonzero element x of a polyhedral cone $C \subseteq \mathbb{R}^n$ is called an* extreme ray *if there are $n-1$ linearly independent constraints binding at x.*
2. *An extreme ray of the recession cone associated with a polyhedron $\mathcal{P}$ is also called an* extreme ray of $\mathcal{P}$*.*

- Note that if d is an extreme ray, then so is λd for all $\lambda \geq 0$.
- Two extreme rays are *equivalent* if one is a multiple of the other.
- When we consider the set of all extreme rays, we will only consider one ray from each equivalence class.
- Note that a polyhedral cone has a finite number of "non-equivalent" extreme rays.

Optimizing Over Pointed Cones

Theorem 2. *Consider the problem of minimizing* $c^\top x$ *over a pointed polyhedral cone* C*. The optimal cost is* $-\infty$ *if and only if some extreme ray* d *of* C *satisfies* $c^\top d < 0$*.*

Proof:

Characterizing Unbounded LPs

Theorem 3. *Consider the LP* $\min\{c^\top x | Ax \geq b\}$ *and assume the feasible region has at least one extreme point. The optimal cost is equal to* $-\infty$ *if and only if some extreme ray* d *satisfies* $c^\top d < 0$.

Proof:

Unboundedness in the Simplex Method

- If we have a standard form problem which is unbounded, the simplex algorithm provides an extreme ray satisfying $c^\top d < 0$.
- When simplex terminates, there is a column j with negative reduced cost and for which basic direction j belongs to the recession cone.
- It is easy to show that this basic direction is an extreme ray of the recession cone.

Representation of Polyhedra

Theorem 4. *Let $\mathcal{P} = \{x \in \mathbb{R}^n\}$ be a nonempty polyhedron with at least one extreme point. Let $x^1, \ldots, x^k$ be the extreme points and $w^1, \ldots, w^r$ be the extreme rays. Then*

$$P = \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \mid \lambda_i \geq 0, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Proof:

Corollaries to the Representation Theorem

Corollary 1. *A nonempty bounded polyhedron, is the convex hull of its extreme points.*

Corollary 2. *A nonempty polyhedron is bounded if and only if it has no extreme rays.*

Corollary 3. *Every element of a polyhedral cone can be expressed as a nonnegative linear combination of extreme rays.*

The Converse of the Representation Theorem

Definition 4. *A set* Q *is* finitely generated *if it is of the form*

$$P = \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \mid \lambda_i \geq 0, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

for given vectors $x^1, \ldots, x^k$ *and* $w^1, \ldots, w^r$ *in* $\mathbb{R}^n$.

Theorem 5. *Every finitely generated set is a polyhedron. The convex hull of finitely many vectors is a bounded polyhedron, also called a* polytope.