Introduction to Mathematical Programming IE406

Lecture 11

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Reading for This Lecture

- Bertsimas 4.4-4.6

More on Complementary Slackness

- Recall the complementary slackness conditions,

$$\begin{aligned} p^\top (Ax - b) &= 0, \\ (c^\top - p^\top A)x &= 0. \end{aligned}$$

- If the primal is in standard form, then any feasible primal solution satisfies the first condition.
- If the dual is in standard form, then any feasible dual solution satisfies the second condition.
- Typically, we only need to worry about satisfying the second condition, which is enforced by the simplex method.

Dual Variables and Marginal Costs

- Consider an LP in standard form with a nondegenerate, optimal basic feasible solution x^* and optimal basis B.
- Suppose we wish to perturb the right hand side slightly by replacing b with $b + d$.
- As long as d is "small enough," we have $B^{-1}(b + d) > 0$ and B is still an optimal basis.
- The optimal cost of the perturbed problem is

$$c_B^\top B^{-1}(b + d) = p^\top (b + d)$$

- This means that the optimal cost changes by $p^\top d$.
- Hence, we can interpret the optimal dual prices as the marginal cost of changing the right hand side of the i^{th} equation.

Economic Interpretation

- The dual prices, or *shadow prices* can allow us to put a value on resources.
- Consider the simple product mix problem from Lecture 9.
- By examining the dual variable for the production hours constraint, we can determine the value of an extra hour of production time.
- We can also determine the maximum amount we would be willing to pay to borrow extra cash.
- Note that the reduced costs can be thought of as the shadow prices associated with the nonnegativity constraints.

Economic Interpretation of Optimality

- Consider again the product mix example from the Lecture 9.
- Using the shadow prices, we can determine how much each product "costs" in terms of its constituent resources.
- The reduced cost of a product is the difference between its selling price and the (implicit) cost of the constituent resources.
- If we discover a product whose "cost" is less than its selling price, we try to manufacture more of that product to increase profit.
- With the new product mix, the demand for various resources is changed and their prices are adjusted.
- We continue until there is no product with cost less than its selling price.
- This is the same as having the reduced costs nonpositive (recall this was a maximization problem).
- Complementary slackness says that we should only manufacture products for which cost and selling price are equal.
- This can be viewed as a sort of multi-round auction.

Shadow Prices in AMPL

Again, recall the model from Lecture 9.

```
ampl: model simple.mod
ampl: solve;
CPLEX 7.0.0: optimal solution; objective 105000
2 simplex iterations (0 in phase I)
ampl: display hours;
hours = 0.5
```

- This tells us that the optimal dual value of the hours constraint is 0.5.
- Increasing the hours by 2000 will increase profit by $(2000)(0.5) = \$1000$.
- Hence, we should be willing to pay up to \$.50/hour for additional hours (as long as the solution remains feasible).

The Dual Simplex Method

- We now present a dual version of the simplex method in tableau form.
- Recall the simplex tableau

$-c_B^\top x_B$	$\bar{c}_1$	$\cdots$	$\bar{c}_n$
$x_{B(1)}$ $\vdots$ $x_{B(m)}$	$B^{-1}A_1$	$\cdots$	$B^{-1}A_n$

- In the dual simplex method, the basic variables are allowed to take on negative values, but we keep the reduced costs nonnegative.

Choosing the Pivot Element

- The pivot row is any row in which the value of the basic variable is negative.
- To determine the pivot column, we perform a ratio test.
- The ratio test determines the largest step length that will maintain dual feasibility, i.e., keep the reduced costs nonnegative.
- Consider the pivot row v—if $v_i \geq 0 \ \forall i$, then the optimal dual cost is $+\infty$ (the primal problem is infeasible).
- Otherwise, if $v_i < 0$, compute the ratio $-\frac{\bar{c}_i}{v_i}$.
- The pivot column is one of the columns with the minimum ratio.
- Pivoting is done in exactly the same way as before.

Comments on Dual Simplex

- Note that a given basis determines both a unique solution to the primal and a unique solution to the dual.

$$\begin{aligned} x_B &= B^{-1}b \\ p^\top &= c_B^\top B^{-1} \end{aligned}$$

- Both the primal and dual solutions are basic and either one, or both, may be feasible.
- If they are both feasible, then they are both optimal.
- Both versions of the simplex method go from one adjacent basic solution to another until reaching optimality.
- Both versions either terminate in a finite number of steps or cycle.
- The dual simplex method is not exactly the same as the simplex method applied to the dual.

Why Use Dual Simplex

- Note that when we can't find a primal feasible basis, we may be able to find a dual feasible basis.
- For a primal problem in standard form with nonnegative costs, we always have a dual feasible solution.
- Suppose we have an optimal basis and we change the right hand side so that the basis becomes primal infeasible.
- The basis will still be dual feasible and so we can continue on with the dual simplex method.
- Note that we can switch back and forth between the two methods.

Dual Degeneracy

- Consider an LP in standard form.
- Recall that the reduced costs are the slack in the dual constraints.
- The reduced costs that are zero correspond to binding dual constraints.
- A dual solution is degenerate if and only if the reduced cost of some nonbasic variable is zero.
- Primal and dual degeneracy are not connected—two bases can lead to the same primal solution, but different dual solutions and vice versa.
- Two bases can even lead to the same primal solution and different dual solutions, one of which is feasible and the other of which is not.
- Dual degeneracy can also cause problems.

Geometric Interpretation of Optimality

- Suppose we have a problem in inequality form, so that the dual is in standard form, and a basis B.
- If I is the index set of binding constraints at the corresponding (nondegenerate) BFS, and we enforce complementary slackness, then dual feasibility is equivalent to

$$\sum_{i \in I} p_i a_i = c.$$

- In other words, the objective function must be a nonnegative combination of the binding constraints.
- We can easily picture this graphically.

Farkas' Lemma

Proposition 1. *Let* $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$ *be given. Then exactly one of the following holds:*

1. $\exists x \geq 0$ *such that* $Ax = b$.
2. $\exists p$ *such that* $p^\top A \geq 0^\top$ *and* $p^\top b < 0$.

- This is closely related to the geometric interpretation of optimality just discussed.
- There are many equivalent version of Farkas' Lemma from which we can derive optimality conditions.
- Note that when the dual simplex algorithm stops because of infeasibility, then the pivot row provides a proof.

An Asset Pricing Model

- Suppose we are in a market that operates for one period and in which n different assets are traded.
- At the end of the period, the market can be in m different possible states.
- Each asset i has a given price p_i at the beginning of the period.
- We have a payoff matrix R which determines the price r_{si} of asset i at the end of the period if the market is in state s.
- Note that we are allowed to *sell short*, which means selling some quantity of asset i at the beginning of the period and buying it back at the end.
- Asset pricing models typically try to determine prices for which there are no *arbitrage opportunities*.
- This means there is no portfolio with a negative cost, but a positive return *in every state*.

Applying Linear Programming

- We can develop a linear program to look for arbitrage opportunities.
- Suppose we let the vector x represent our portfolio at the beginning of the period.
- The condition that our return should be positive in every state is simply

$$Rx \geq 0$$

- The condition that the portfolio has negative cost is simply

$$p^\top x \leq 0$$

- Hence, we can simply solve the LP $\min\{p^\top x | Rx \geq 0\}$.

Asset Pricing Using Farkas' Lemma

- The absence of arbitrage is equivalent to the condition that $Rx \geq 0 \Rightarrow p^\top x \geq 0$.
- This is the same as the LP above have a nonnegative optimal solution.
- By Farkas' Lemma, the absence of arbitrage opportunities is equivalent to the existence of a vector of nonnegative *state prices* q such that

$$p = q^\top R$$

- Hence, if we determine such state prices and use them to value existing assets, we eliminate the possibility of arbitrage.
- This is a key concept in modern finance theory.