Introduction to Mathematical Programming IE406

Lecture 10

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Reading for This Lecture

- Bertsimas 4.1-4.3

Duality Theory: Motivation

- Consider the following minimization problem

$$\min x^2 + y^2$$
$$s.t.\ \ x + y = 1$$

- How could we solve this problem?
- Idea: Consider the function

$$L(x, y, p) = x^2 + y^2 + p(1 - x - y)$$

- What can we do with this?

Lagrange Multipliers

- The idea is not to strictly enforce the constraints.
- We associate a Lagrange multiplier, or *price*, with each constraint.
- Then we allow the constraint to be violated *for a price*.
- Consider an LP in standard form.
- Using Lagrange multipliers, we can formulate an alternative LP:

$$\begin{aligned} &\min\ c^\top x + p^\top (b - Ax) \\ &s.t.\quad x \geq 0 \end{aligned}$$

- How does the optimal solution of this compare to the original optimum?

Lagrange Multipliers

- Because we haven't changed the cost of feasible solutions to the original problem, this new problem gives a lower bound.

$$g(p) = \min_{x \geq 0} \left[c^\top x + p^\top (b - Ax) \right] \leq c^\top x^* + p^\top (b - Ax^*) = c^\top x^*$$

- Since each value of p gives a lower bound, we consider maximizing $g(p)$.
- Think of this as finding the *best* lower bound.
- This is known as the *dual problem*.

Simplifying

- In linear programming, we can obtain an explicit form for the dual.

$$\begin{aligned} g(p) &= min_{x \geq 0} \left[c^\top x + p^\top (b - Ax)\right] \\ &= p^\top b + min_{x \geq 0}(c^\top - p^\top A)x \end{aligned}$$

- Note that

$$min_{x \geq 0}(c^\top - p^\top A)x = \begin{cases} 0, & \text{if } c^\top - p^\top A \geq \mathbf{0}^\top, \\ -\infty, & \text{otherwise,} \end{cases}$$

- Hence, we can show that the dual is equivalent to

$$\begin{aligned} &\max\ p^\top b \\ &s.t. \quad p^\top A \leq c^\top \end{aligned}$$

Inequality Form

- Suppose our feasible region is $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \geq b, x \geq 0\}$.
- We can add slack variables and convert to standard form with constraints

$$[A| - I] \begin{bmatrix} x \\ s \end{bmatrix} = b$$

- This leads to dual constraints

$$p^\top [A| - I] \leq [c^\top | \mathbf{0}^\top]$$

- Hence, we get the dual

$$\begin{aligned} \max\ & p^\top b \\ s.t.\ & p^\top A \leq c^\top \\ & p \geq 0 \end{aligned}$$

From the Primal to the Dual

We can dualize general LPs as follows

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Properties of the Dual

- All equivalent forms of the primal give equivalent forms of the dual.
- The dual of the dual is the primal.
- Weak Duality: If x is a feasible solution to the primal and p is a feasible solution to the dual, then

$$p^\top b \leq c^\top x$$

- Corollaries:
  - If the optimal cost of the primal is $-\infty$, then the dual is infeasible.
  - If the optimal cost of the dual is $+\infty$, then the primal is infeasible.
  - If x is a feasible primal solution and p is a feasible dual solution such that $c^\top x = p^\top b$, then both x and p are optimal.

Relationship of the Primal and the Dual

The following are the possible relationships between the primal and the dual:

	Finite Optimum	**Unbounded**	**Infeasible**
Finite Optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Strong Duality

Proposition 1. *(Strong Duality) If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.*

Proof:

More About the Dual

- When we interpret the quantity $c_B^\top B^{-1}$ as the vector of dual prices, the reduced costs are then the slack in the constraints of the dual.
- The condition that all the reduced costs be nonnegative is then equivalent to dual feasibility.
- Hence, the simplex algorithm can be interpreted as maintaining primal feasibility while trying to achieve dual feasibility.
- We will shortly see an alternative algorithm which maintains dual feasibility while trying to achieve primal feasibility.

Complementary Slackness

Proposition 2. *If x and p are feasible primal and dual solutions to a general linear program with constraint matrix $A \in \mathbb{R}^{m \times n}$ and right-hand side vector $b \in \mathbb{R}^m$, then x and p are optimal if and only if*

$$\begin{aligned} p^\top (Ax - b) &= 0, \\ (c^\top - p^\top A)x &= 0. \end{aligned}$$

Proof:

Optimality Without Simplex

Let's consider an LP in standard form. We have now shown that the optimality conditions for (nondegenerate) x are

1. $Ax = b$ (primal feasibility)
2. $x \geq 0$ (primal feasibility)
3. $x_i = 0$ if $p^\top a_i \neq c_i$ (complementary slackness)
4. $p^\top A \leq c$ (dual feasibility)

- In standard form, the complementary slackness condition is simply $x^\top \bar{c} = 0$.
- This condition is always satisfied during the simplex algorithm, since the reduced costs of the basic variables are zero.