Advanced Operations Research Techniques IE316

Lecture 5

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Reading for This Lecture

- Bertsimas 2.5-2.7

Existence of Extreme Points

Definition 1. *A polyhedron* $\mathcal{P} \in \mathbb{R}^n$ contains a line *if there exists a vector* $x \in \mathcal{P}$ *and a nonzero vector* $d \in \mathbb{R}^n$ *such that* $x + \lambda d \in \mathcal{P} \ \forall \lambda \in \mathbb{R}$.

Theorem 1. *Suppose that the polyhedron* $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \geq b\}$ *is nonempty. Then the following are equivalent:*

- *The polyhedron* $\mathcal{P}$ *has at least one extreme point.*
- *The polyhedron* $\mathcal{P}$ *does not contain a line.*
- *There exist* n *rows of* A *that are linearly independent.*

Optimality of Extreme Points

Theorem 2. *Let* $\mathcal{P} \subseteq \mathbb{R}^n$ *be a polyhedron and consider the problem* $\min_{x \in \mathcal{P}} c^T x$ *for a given* $c \in \mathbb{R}^n$*. If* $\mathcal{P}$ *has at least one extreme point and there exists an optimal solution, then there exists an optimal solution that is an extreme point.*

Proof:

Optimality in Linear Programming

- For linear optimization, a finite optimal cost is equivalent to the existence of an optimal solution.
- The previous result can be strengthened.
- Since any linear programming problem can be written in standard form, we can derive the following result:

Theorem 3. *Consider the linear programming problem of minimizing* $c^T x$ *over a nonempty polyhedron. Then, either the optimal cost is* $-\infty$ *or there exists an optimal solution which is an extreme point.*

Representation of Polyhedra

Theorem 4. *A nonempty, bounded polyhedron is the convex hull of its extreme points.*

Theorem 5. *The convex hull of a finite set of vectors is a polyhedron.*

Notes:

Example: Product Mix

- In this example, we consider Top Brass Trophy, a shop that manufactures two kinds of trophies, football and soccer.
- Resource requirements
  - Football trophies: 1 brass football, 1 plaque, 4 board feet of wood.
  - Soccer Trophies: 1 brass soccer ball, 1 plaque, 2 board feet of wood.
- Resource constraints
  - 1000 footballs
  - 1500 soccer balls
  - 1750 plaques
  - 4800 board feet of wood
- Profit is $\$12$ on football trophies and $\$9$ on soccer trophies.
- The goal is to maximize profit.

Top Brass Example: Formulation

- What are the decision variables?
- What is the objective function?
- What are the constraints?

Top Brass Example: Solving

- Basic scheme
  - Rewrite constraints in standard form.
  - Find an initial basic feasible solution.
  - Move to an adjacent vertex that improves the solution value.
  - Keep moving until no further improvement is possible.
- Question: What sets of variables do not form a basis?

Top Brass Example: Degeneracy

- Suppose we had the additional constraint $3x_1 + 2x_2 \leq 5000$.
- Note that this constraint is *redundant*.
- This new constraint is linearly dependent on the other constraints.
- Initially, we may not know this.
- What could happen to our solution method?

Top Brass Example: Alternative Representation

- Denote the polyhedron from the example by $\mathcal{P}$.
- Denote the extreme points of $\mathcal{P}$ by $p_1, \ldots, p_6$.
- Then we could also represent $\mathcal{P}$ as

$$\mathcal{P} = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{6} \lambda_i p_i, \sum_{i=1}^{6} \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, 6\}$$

- Rewriting the objective function as

$$\max c^T \sum_{i=1}^{6} \lambda_i p_i,$$

we have another form of the LP.

What We've Learned So Far

- We are interested in the extreme points of polyhedra.
- There is a one-to-one correspondence between the extreme points of a polyhedron and the basic feasible solutions.
- We can construct basic solutions by
  - Choosing a basis B of m linearly independent columns of A.
  - Solve the system $Bx_B = b$ to obtain the values of the basic variables.
  - Set $x_N = 0$.
- We can move between adjacent (nondegenerate) basic solutions by removing one column of the basis and replacing it with another.
- In the presence of degeneracy, we might stay at the same extreme point.
- These are the building blocks we need to construct algorithms for solving LPs.