Advanced Operations Research Techniques IE316

Lecture 11

Dr. Ted Ralphs

Reading for This Lecture

- Bertsimas 4.1-4.3

Duality Theory: Motivation

- Consider the following minimization problem

$$\begin{aligned} &minimize\, x^2 + y^2 \\ &s.t. \qquad\quad x + y = 1 \end{aligned}$$

- How could we solve this problem?
- Idea: Consider the function

$$L(x, y, p) = x^2 + y^2 + p(1 - x - y)$$

- What can we do with this?

Lagrange Multipliers

- The idea is not to strictly enforce the constraints.
- We associated a Lagrange multiplier, or *price*, with each constraint.
- Then we allow the constraint to be violated *for a price*.
- Consider an LP in standard form.
- Using Lagrange multipliers, we can formulate an alternative LP:

$$\begin{array}{ll} minimize & c^T x + p^T(b - Ax) \\ s.t. & x \geq 0 \end{array}$$

- How does the optimal solution of this compare to the original optimum?

Lagrange Multipliers

- Because we haven't changed the cost of feasible solutions to the original problem, this new problem gives a lower bound.

$$g(p) = \min_{x \geq 0} \left[c^T x + p^T(b - Ax)\right] \leq c^T x^* + p^T(b - Ax^*) = c^T x^*$$

- Since each value of p gives a lower bound, we consider maximizing $g(p)$.
- Think of this as finding the *best* lower bound.
- This is known as the *dual problem*.

Simplifying

- In linear programming, we can obtain an explicit form for the dual.

$$\begin{aligned} g(p) &= min_{x \geq 0} \left[c^T x + p^T (b - Ax) \right] \\ &= p^T b + min_{x \geq 0} (c^T - p^T A) x \end{aligned}$$

- Note that

$$min_{x \geq 0} (c^T - p^T A) x = \begin{cases} 0, & \text{if } c^T - p^T A \geq \mathbf{0}^T, \\ -\infty, & \text{otherwise,} \end{cases}$$

- Hence, we can show that the dual is equivalent to

$$\begin{aligned} &maximize \;\; p^T b \\ &s.t. \qquad\quad p^T A \leq c^T \end{aligned}$$

Inequality Form

- Suppose we have an LP in inequality form.
- We can add slack variables and convert to standard form with constraints

$$[A|-I]\begin{bmatrix} x \\ s \end{bmatrix} = b$$

- This leads to dual constraints

$$p^T\,[A|-I] \leq \left[c^T|\mathbf{0}^T\right]$$

- Hence, we get the dual

$$\begin{array}{ll} maximize & p^T b \\ s.t. & p^T A \leq c^T \\ & p \geq 0 \end{array}$$

From the Primal to the Dual

We can dualize general LPs as follows

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Properties of the Dual

- All equivalent forms of the primal give equivalent forms of the dual.
- The dual of the dual is the primal.
- Weak Duality: If x is a feasible solution to the primal and p is a feasible solution to the dual, then

$$p^T b \leq c^T x$$

- Corollaries:
  - If the optimal cost of the primal is $-\infty$, then the dual is infeasible.
  - If the optimal cost of the dual is $+\infty$, then the primal is infeasible.
  - If x is a feasible primal solution and p is a feasible dual solution such that $c^T x = p^T b$, then both x and p are optimal.

Relationship of the Primal and the Dual

The following are the possible relationships between the primal and the dual:

	Finite Optimum	**Unbounded**	**Infeasible**
Finite Optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Strong Duality

Proposition 1. *(Strong Duality) If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.*

Proof:

More About the Dual

- When we interpret the quantity $c_B B^{-1}$ as the vector of dual prices, the reduced costs are then the slack in the constraints of the dual.
- The condition that all the reduced costs be nonnegative is then equivalent to dual feasibility.
- Hence, the simplex algorithm can be interpreted as maintaining primal feasibility while trying to achieve dual feasibility.
- We will shortly see an alternative algorithm which maintains dual feasibility while trying to achieve primal feasibility.

Complementary Slackness

Proposition 2. *If x and p are feasible primal and dual solutions, then x and p are optimal if and only if*

$$\begin{aligned} p^T(Ax - b) &= 0, \\ (c^T - p^T A)x &= 0. \end{aligned}$$

Proof:

Optimality Without Simplex

Let's consider an LP in standard form. We have now shown that the optimality conditions for (nondegenerate) x are

1. $Ax = b$ (primal feasibility)
2. $x \geq 0$ (primal feasibility)
3. $x_i = 0$ if $p^T a_i < c_i$ (complementary slackness)
4. $p^T A \leq c$ (dual feasibility)

- In standard form, the complementary slackness condition is simply $x^T \bar{c} = 0$.
- This condition is always satisfied during the simplex algorithm, since the reduced costs of the basic variables are zero.