Algorithms in Systems Engineering
ISE 172

Lecture 21

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References for Today's Lecture

- Required reading
  - CLRS Chapter 23
- References
  - R. Sedgewick, *Algorithms in C++* (Third Edition), 1998.

Spanning Trees

- Given a connected undirected graph $G = (V, E)$, a *spanning tree* T of G is a subgraph that is a tree and whose vertex set is all of V.
- Since the vertex set of any such spanning tree is V, we will sometimes equate the edge set of a spanning tree with the spanning tree itself.
- Every minimal connected subgraph is a spanning tree (and vice versa).
- In other words, a subgraph is a spanning tree if and only if it is connected and removing any edge will disconnect it.
- If we are looking for the most inexpensive set of links that connect a set of geographically dispersed points, we want a spanning tree.
- Spanning trees arise frequently in applications, especially those with a network design component.
- They also arise in other applications, such as the design of integrated circuits.

Minimum Weight Spanning Trees

- Consider an undirected graph $G = (V, E)$ with weight vector $w \in \mathbb{R}^E$.
- If $T \subseteq E$ is a spanning tree of G, the *weight* of T is

$$\sum_{e \in T} w_e$$

- The *minimum weight spanning tree* (MST) problem is that of finding, among all spanning trees of G, one that has minimum weight.
- How many spanning tree are there?
- What about simply enumerating all of them?

Finding a Minimum Weight Spanning Tree

- Although finding an MST may seem to be a difficult problem, it can be solved efficiently.
- The first algorithm we'll consider uses another variant of graph search to build a search tree that is guaranteed to be an MST.
- We build the tree up, adding one vertex at a time.
- At each iteration, we have a partially completed tree that spans the vertices that have been processed so far.
- The vertex that is processed next is the one that is "closest" to the partially completed tree.
- The algorithm is almost identical to Dijkstra's Algorithm.

Algorithm Summary: Prim's Algorithm

- We are given a connected undirected weighted graph $G = (V, E)$ and we want to find an MST of G.
- Prim's Algorithm
  - Arbitrarily choose a source node r.
  - Initialize by assigning $d(r) = 0$ for the source node and $d(v) = \infty$ for all other nodes $v \in V \setminus \{r\}$.
  - Place r on the list L of unprocessed nodes.
  - While L is not empty
    * Choose $v \in L$ such that $d(v) = \min_{u \in L} d(u)$.
    * For each neighbor x of v, set $d(x) = \min\{d(x), w_{\{v,x\}}\}$.
- When we're finished, the search tree will be an MST.
- Why is this algorithm correct?
- How do we implement it?
- What is the running time?

Another View of Prim's Algorithm

- Prim's Algorithm can be viewed as a special case of graph search.
- The algorithm can also be viewed as a special case of another general class of algorithms called *greedy algorithms*.
- A *greedy algorithm* is one that makes the choice at each step that looks the best "at the moment" and doesn't reconsider that choice later.
- We can view the construction of an MST as a greedy algorithm, but first we must define some terminology.
- Given an undirected graph $G = (V, E)$, a *cut* is a set $S \subset V$ that defines a partition of V into two nonempty subsets, S and $V \setminus S$.
- An edge is said to *cross the cut* if it connects a node in S to a node in $V \setminus S$.
- Our goal is to build a spanning tree by adding one edge at a time to a set T in a "greedy" fashion.
- Basically, we just need to somehow guarantee ourselves that at each step, the current set can be "extended" to an MST.
- How do we do that?

Safe Edges

- Let's assume that our current set of edges T already satisfies the property that T can be extended to an MST.
- Question: What edges can we add to T to maintain the property?
- Answer: Any edge that is a minimum edge crossing some cut S.
- Rationale: In any connected graph, there must be an edge crossing each cut in the graph (why?)
- We will call such edge a *safe edge* if it also doesn't create a cycle when added to T.
- How do we find such an edge?
  - Prim's Algorithm simply considers the cut S consisting of nodes that have already been processed.
  - At each step, we add the minimum edge crossing that cut.
  - There are other possibilities, however.

Generic Greedy Algorithm for Building an MST

- Generic greedy algorithm for constructing a spanning tree.
  - Set $T = \emptyset$.
  - Select a safe edge and add it to T.
  - Repeat until T is a spanning tree.
- This is guaranteed to work, no matter how the safe edges are selected.