Solving Symmetric Integer Programs

Jim Ostrowski
ISE Department
COR@L Lab
Lehigh University
jao204@lehigh.edu
Some Preliminaries

- For a set $S \subseteq \mathbb{I}^n$, the orbit of $S$ with respect to $\Gamma$ is the set of all subsets of $\mathbb{I}^n$ to which $S$ can be sent by permutations in $\Gamma$:

  $$\text{orb}(S, \Gamma) \overset{\text{def}}{=} \{ S' \subseteq \mathbb{I}^n \mid \exists \pi \in \Gamma \text{ such that } S' = \pi(S) \}.$$  

- I care (mostly) about orbits of sets of cardinality one, corresponding to decision variables $x_j$.

- By definition, if $j \in \text{orb}(\{k\}, \Gamma)$, then $k \in \text{orb}(\{j\}, \Gamma)$, i.e. the variable $x_j$ and $x_k$ share the same orbit. Therefore, the union of the orbits

  $$\mathcal{O}(\Gamma) \overset{\text{def}}{=} \bigcup_{j=1}^{n} \text{orb}(\{j\}, \Gamma)$$

  forms a partition of $\mathbb{I}^n = \{1, 2, \ldots, n\}$, which we refer to as the orbits of $\Gamma$.

- The orbits encode which variables are “equivalent” with respect to the symmetry $\Gamma$. 

Jim Ostrowski  (Lehigh University)  Solving Symmetric IPs  Miami University
Let $O \in \mathcal{C}(A)$ be an orbit of the symmetry group of $A$ representing constraints, $h$ any element in $O$.

Surely we can branch on the disjunction

$$c_h x = b \lor \{c_j x \geq b + 1 | \forall j \in O\}$$
Basic Idea

- So now I have subproblems with equalities in them...
- Use them!
- Create a relaxation to the subproblem by removing all variables not included in a chosen equality constraint and remove all constraints which include a removed variable.
- Find the collection of all non-isomorphic solutions to the relaxation.
- Use these solutions as partial solutions to the original subproblem, branch on these solutions.
Steiner Triple Systems

- Let $X$ be a set of $v \geq 3$ elements.
- $B$ is collection of 3 elements subsets of $X$ s.t. every pair of elements of $X$ is found in exactly one element of $B$.
- $S_3 = \{\{1, 2, 3\}\}$
- $S_7 = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$
- A Steiner Triple System exists if $v = 1, 3 \mod(6)$. 
Growing Triple Systems

- Let's build $S_9$ using $S_3 = \{1, 2, 3\}$.
Growing Triple Systems

- Lets build $S_{9}$ using $S_{3} = \{1, 2, 3\}$.
- First, start with 3 $S_{3}$'s, $X_{1} = \{1, 2, 3\}$, $X_{2} = \{4, 5, 6\}$, $X_{3} = \{7, 8, 9\}$ and their corresponding triple $\{\{1, 2, 3\}\}, \{\{4, 5, 6\}\}, \{\{7, 8, 9\}\}$.
- Link “like elements” with sets $\{1, 4, 7\}$, $\{2, 5, 8\}$, $\{3, 6, 9\}$ (i.e. form a set with all the first elements of each $X$, second elements, ...).
- Link remaining elements using solution to $S_{3}$. Using $\{\{1, 2, 3\}\}$, create sets by choosing the first element from set $X_{i}$, the second element from set $X_{j}$ ($j \neq i$), and the third element from set $X_{k}$ ($k \neq i, j$).
Growing Triple Systems

\[ A_9 = \begin{bmatrix}
1_{s_1} & 2_{s_1} & 3_{s_1} & 1_{s_2} & 2_{s_2} & 3_{s_2} & 1_{s_3} & 2_{s_3} & 3_{s_3} \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix} \]
A Hard Integer Program

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{7} x_i \\
\text{s.t.} & \quad x_1 + x_2 + x_4 \geq 1 \\
& \quad x_2 + x_3 + x_5 \geq 1 \\
& \quad x_3 + x_4 + x_6 \geq 1 \\
& \quad x_4 + x_5 + x_7 \geq 1 \\
& \quad x_1 + x_5 + x_6 \geq 1 \\
& \quad x_2 + x_6 + x_7 \geq 1 \\
& \quad x_7 + x_1 + x_3 \geq 1 \\
\end{align*}
\]

\[x \in \{0, 1\}^n\]
A Hard Integer Program

- For sts7, for every triple \( \{a, b, c\} \) in \( S_7 \), create a constraint
  \[
  x_a + x_b + x_c \geq 1
  \]

- These problems are very difficult.
- The smallest unsolved instance has only 135 variables.
- Why are these problems so difficult?
- These problems have a large degree of symmetry, but that is not all.
- The LP relaxation of sts135 is 45, the smallest known feasible solution has value 103.
- A large gap between the relaxation and the optimal solution make integer programs very hard to solve.
Improving the Gap

- sts135 was created by using 3 sts45 problems (implying there is symmetry present in the problem)
- The optimal solution of sts45 is 30, so...
Improving the Gap

- sts135 was created by using 3 sts45 problems (implying there is symmetry present in the problem).
- The optimal solution of sts45 is 30, so...
- We know that $\sum_{i=1}^{45} x_i \geq 30$, $\sum_{i=46}^{90} x_i \geq 30$, and $\sum_{i=91}^{135} x_i \geq 30$.
- Adding these inequalities increases the LP relaxation to 90.
- Nice, but we can do better!
Exploiting Symmetry of Constraints

- Similar to variables, constraints can be symmetric.
- We can use orbital branching to “branch” on the constraints.
Exploiting Symmetry of Constraints

- Similar to variables, constraints can be symmetric.
- We can use orbital branching to “branch” on the constraints.
- So... Either \( \sum_{i=1}^{45} x_i = 30 \) or \( \sum_{i=1}^{45} x_i \geq 31 \), \( \sum_{i=46}^{90} x_i \geq 31 \), and \( \sum_{i=91}^{135} x_i \geq 31 \)
- So what good does this do?
But Wait, There is More!

- How many sts45’s are in an sts135?
- The constraint \( \sum_{i=1}^{45} x_i \geq 30 \) is also equivalent to the constraints:
Steiner Triple Systems

\[ \sum_{i=1}^{15} x_i + \sum_{i=46}^{60} x_i + \sum_{i=91}^{105} x_i \geq 30 \]

\[ \sum_{i=1}^{15} x_i + \sum_{i=61}^{75} x_i + \sum_{i=121}^{135} x_i \geq 30 \]

\[ \sum_{i=1}^{15} x_i + \sum_{i=76}^{90} x_i + \sum_{i=106}^{120} x_i \geq 30 \]

\[ \sum_{i=16}^{30} x_i + \sum_{i=46}^{60} x_i + \sum_{i=121}^{135} x_i \geq 30 \]

\[ \sum_{i=16}^{30} x_i + \sum_{i=61}^{75} x_i + \sum_{i=106}^{120} x_i \geq 30 \]

\[ \sum_{i=16}^{30} x_i + \sum_{i=76}^{90} x_i + \sum_{i=91}^{105} x_i \geq 30 \]
Exploiting Symmetry of Constraints

- Using orbital branching we can generate all non-isomorphic solutions to sts45 with value 30
- (There is only 1 of them)
- The subproblem formed by setting \( \sum_{i=1}^{45} x_i = 30 \) can solved by fixing the first 45 variables to correspond to the solution of sts45.
- Lather, rinse, repeat...
- Keep branching on constraints until we increase rhs to 34
- Why? With the rhs of 35, the LP relaxation is 105, so these problems cannot contain a solution of size 103 or better
How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30
How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30
- 246 solutions of sts45 of value 31
How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30
- 246 solutions of sts45 of value 31
- 9497 solutions of sts45 of value 32
How many subproblems are there

- There are...
- 2 solutions of sts45 of value 30
- 246 solutions of sts45 of value 31
- 9497 solutions of sts45 of value 32
- 61539 solutions of sts45 of value 33
How many subproblems are there

- There are...
- 2 solutions of $sts45$ of value 30
- 246 solutions of $sts45$ of value 31
- 9497 solutions of $sts45$ of value 32
- 61539 solutions of $sts45$ of value 33
- 122972 solutions of $sts45$ of value 34
- ... CRAP! ...
But are all of them non-isomorphic?

- NO! By checking for isomorphism we can remove...
But are all of them non-isomorphic?

- NO! By checking for isomorphism we can remove...
- 13 of the solutions for sts45 of values 30-32...
- Yippee, I saved 2 hours of computation time!!!
But are all of them non-isomorphic?

- NO! By checking for isomorphism we can remove...
- 13 of the solutions for sts45 of values 30-32...
- Yippee, I saved 2 hours of computation time!!!
- I can do much, much better!
Exploiting Symmetry of Constraints

- **Pros:** Each subproblem will have 45 variables fixed and are much easier to solve.
- This can be done easily in parallel, different computers can solve different subproblems independently of each other.
- **Cons:** In order to solve $sts_{135}$ we need to generate all non-isomorphic solutions to $sts_{45}$ with values 30-34.
- There can be a whole lot of these.
- Generating these may take awhile.