A Different Perspective on Perspective Cuts

JEFF LINDEROTH
Unemployed
linderot@cs.wisc.edu

OKTAY GÜNLÜK
Mathematical Sciences Department
IBM T.J. Watson Research Center
gunluk@us.ibm.com

MIP 2007
Centre de recherches mathématiques
Université de Montréal
August 1, 2007
Indicator MINLPs

- We focus on (convex) MINLPs that are driven by 0-1 indicator variables $z_i, i \in \mathcal{I}$
- Each indicator variable $i$ controls a collection of variables $V_i$
- If $z_i = 0$, the components of $x$ controlled by $z_i$ must collapse to a point: $z_i = 0 \Rightarrow x_{V_i} = \hat{x}_{V_i}$
  - WLOG $\hat{x}_{V_i} = 0$ from now on
- If $z_i = 1$, the components of $x$ controlled by $z_i$ belong to a convex set $z_i = 1 \Rightarrow x_{V_i} \in \Gamma_i$
- $\Gamma_i$ is specified by (convex) nonlinear inequality constraints and bounds on the variables

$$\Gamma_i \overset{\text{def}}{=} \{x_{V_i} \mid f_k(x_{V_i}) \leq 0 \ \forall k \in K_i, l \leq x_{V_i} \leq u\}.$$
Indicator MINLPs

\[
\begin{align*}
\min & \quad c^T x + d^T z \\
\text{s.t.} & \quad g_m(x, z) \leq 0 \quad \forall m \in M \\
& \quad z_i f_k(x_{V_i}) \leq 0 \quad \forall i \in I \quad \forall k \in K_i \\
& \quad \ell_j z_i \leq x_j \leq u_j z_i \quad \forall i \in I \quad \forall j \in V_i \\
x & \in X \\
z & \in Z \cap B^p, 
\end{align*}
\]

- $X, Z$ polyhedral sets
- Typically, \( g_m(x, z) = \bar{g}_m(x) + a_m^T z \) is linear in $z$, or even $a_m = 0$. 
Indicator MINLPs

\[
\begin{align*}
\min & \quad c^T x + d^T z \\
\text{s.t.} & \quad g_m(x, z) \leq 0 \quad \forall m \in M \\
& \quad z_i f_k(x_V) \leq 0 \quad \forall i \in I \quad \forall k \in K_i \\
& \quad \ell_j z_i \leq x_j \leq u_j z_i \quad \forall i \in I \quad \forall j \in V_i \\
& \quad x \in X \quad z \in Z \cap \mathbb{B}^p,
\end{align*}
\]

- $X, Z$ polyhedral sets
- Typically, $g_m(x, z) = \bar{g}_m(x) + \alpha_m^T z$ is linear in $z$, or even $\alpha_m = 0$.
- If $z \in Z \cap \mathbb{B}^p$ is fixed, then the problem is convex.
Indicators Everywhere

Process Flow Applications

- $z = 0 \implies x_1 = x_2 = x_3 = x_4 = 0$
- $z = 1 \implies f(x_1, x_2, x_3, x_4) \leq 0$

Note that here $I$ is already lying $z = 0$ does not imply $y = 0$ Nevertheless, results apply to epigraph-type indicator MINLPs.
Indicators Everywhere

Process Flow Applications
- $z = 0 \implies x_1 = x_2 = x_3 = x_4 = 0$
- $z = 1 \implies f(x_1, x_2, x_3, x_4) \leq 0$

Separable Function Epigraphs
- $y_i \geq f_i(x_i) \quad \forall i \in I$
- $l z_i \leq x_i \leq u z_i \quad \forall i \in I$
Indicators Everywhere

**Process Flow Applications**
- $z = 0 \Rightarrow x_1 = x_2 = x_3 = x_4 = 0$
- $z = 1 \Rightarrow f(x_1, x_2, x_3, x_4) \leq 0$

**Separable Function Epigraphs**
- $y_i \geq f_i(x_i) \ \forall i \in I$
- $\ell z_i \leq x_i \leq u z_i \ \forall i \in I$

- Note that here I am already lying
- $z = 0$ does not imply $y = 0$
- Nevertheless, results apply to epigraph-type indicator MINLPs.
A Very Simple Example

\[ R \overset{\text{def}}{=} \left\{ (x, y, z) \in \mathbb{R}^2 \times \mathbb{B} \mid y \geq x^2, 0 \leq x \leq uz \right\} \]
A Very Simple Example

\[ R \overset{\text{def}}{=} \left\{ (x, y, z) \in \mathbb{R}^2 \times \mathbb{B} \mid y \geq x^2, 0 \leq x \leq uz \right\} \]

- \( z = 0 \Rightarrow x = 0, y \geq 0 \)
- \( z = 1 \Rightarrow x \leq u, y \geq x^2 \)
A Very Simple Example

\[ R \overset{\text{def}}{=} \left\{ (x, y, z) \in \mathbb{R}^2 \times \mathbb{B} \mid y \geq x^2, 0 \leq x \leq uz \right\} \]

- \( z = 0 \Rightarrow x = 0, y \geq 0 \)
- \( z = 1 \Rightarrow x \leq u, y \geq x^2 \)

Deep Insights

- \( \text{conv}(R) \equiv \text{line connecting } (0, 0, 0) \text{ to } y = x^2 \text{ in the } z = 1 \text{ plane} \)
Characterization of Convex Hull

- Work out the algebra to get:

**Deep Theorem #1**

\[
\text{conv}(R) = \left\{ (x, y, z) \in \mathbb{R}^3 \mid yz \geq x^2, 0 \leq x \leq uz, 0 \leq z \leq 1, y \geq 0 \right\}
\]
Characterization of Convex Hull

- Work out the algebra to get:

Deep Theorem #1

\[ \text{conv}(\mathbb{R}) = \{(x, y, z) \in \mathbb{R}^3 \mid yz \geq x^2, 0 \leq x \leq uz, 0 \leq z \leq 1, y \geq 0\} \]

\[ x^2 \leq yz, y, z \geq 0 \equiv \]
Characterization of Convex Hull

- Work out the algebra to get:

**Deep Theorem #1**

\[
\text{conv}(\mathbb{R}) = \left\{ (x, y, z) \in \mathbb{R}^3 \mid yz \geq x^2, 0 \leq x \leq uz, 0 \leq z \leq 1, y \geq 0 \right\}
\]

\[x^2 \leq yz, y, z \geq 0 \equiv\]

Second Order Cone Programming

- There are effective and robust algorithms for optimizing linear objectives over \(\text{conv}(\mathbb{R})\)
Higher Dimensions

- Using an extended formulation, we can describe the convex hull of a higher-dimensional analogue of $R$:

$$Q \overset{\text{def}}{=} \left\{ (w, x, z) \in \mathbb{R}^{1+n} \times \mathbb{B}^n \mid w \geq \sum_{i=1}^{n} q_i x_i^2, \ u_i z_i \geq x_i \geq 0, \forall i \right\}$$
Higher Dimensions

- Using an extended formulation, we can describe the convex hull of a higher-dimensional analogue of $\mathbb{R}$:

$$Q \overset{\text{def}}{=} \left\{ (w, x, z) \in \mathbb{R}^{1+n} \times \mathbb{B}^n \mid w \geq \sum_{i=1}^{n} q_i x_i^2, \ u_i z_i \geq x_i \geq 0, \ \forall i \right\}$$

- First we write an extended formulation of $Q$, introducing variables $y_i$:

$$\bar{Q} \overset{\text{def}}{=} \left\{ (w, x, y, z) \in \mathbb{R}^{1+3n} \mid w \geq \sum_i q_i y_i, (x_i, y_i, z_i) \in R_i, \ \forall i \right\}$$

$$R_i \overset{\text{def}}{=} \left\{ (x_i, y_i, z_i) \in \mathbb{R}^2 \times \mathbb{B} \mid y_i \geq x_i^2, 0 \leq x_i \leq u_i z_i \right\}$$
Extended Formulations

- $\bar{Q}$ is indeed an extended formulation in the sense that projecting out the $y$ variables from $\bar{Q}$ gives $Q$: $\text{Proj}_{(w,x,z)} \bar{Q} = Q$. 

The convex hull of $\bar{Q}$ is obtained by replacing $R_i$ with its convex hull description $\text{conv}(R_i)$:

$$\text{conv}(\bar{Q}) = \{ w \in \mathbb{R}, x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}^n : w \geq \sum_{i} q_i y_i, (x_i, y_i, z_i) \in \text{conv}(R_i), i = 1, 2, \ldots, n \}.$$ 

Again, the description of $\text{conv}(\bar{Q})$ is SOC-representable.

You get one rotated cone for each $i$. 

Günlük and Linderoth (UW-Madison) A Different Perspective on Perspective Cuts
Extended Formulations

- $\bar{Q}$ is indeed an extended formulation in the sense that projecting out the $y$ variables from $\bar{Q}$ gives $Q$: $\text{Proj}_{(w,x,z)} \bar{Q} = Q$.
- The convex hull of $\bar{Q}$ is obtained by replacing $R_i$ with its convex hull description $\text{conv}(R_i)$:

$$\text{conv}(\bar{Q}) = \left\{ w \in \mathbb{R}, x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}^n : w \geq \sum_i q_i y_i, \right.$$

$$\left. (x_i, y_i, z_i) \in \text{conv}(R_i), \quad i = 1, 2, \ldots, n \right\}.$$
Extended Formulations

- $\bar{Q}$ is indeed an extended formulation in the sense that projecting out the $y$ variables from $\bar{Q}$ gives $Q$: $\text{Proj}(w,x,z) \bar{Q} = Q$.
- The convex hull of $\bar{Q}$ is obtained by replacing $R_i$ with its convex hull description $\text{conv}(R_i)$:

$$\text{conv}(\bar{Q}) = \left\{ w \in \mathbb{R}, \ x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}^n : w \geq \sum_i q_i y_i, \ (x_i, y_i, z_i) \in \text{conv}(R_i), \ i = 1, 2, \ldots, n \right\}.$$

- Again, the description of $\text{conv}(\bar{Q})$ is SOC-representable.
- You get one rotated cone for each $i$. 

Günlük and Linderoth (UW-Madison)
Descriptions in the Original Space

- We can also write a convex hull description in the original space of variables, by projecting out $y$:

$$Q^c = \left\{ (w, x, z) \in \mathbb{R}^{1+n+n} : \right.$$ 

$$w \prod_{i \in S} z_i \geq \sum_{i \in S} \left( q_i x_i^2 \prod_{l \in S \setminus \{i\}} z_l \right) \quad \text{if } S \subseteq \{1, 2, \ldots, n\}$$

$$u_i z_i \geq x_i \geq 0, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n \right\} \tag{\Pi}$$
Descriptions in the Original Space

- We can also write a convex hull description in the original space of variables, by projecting out $y$:

$$Q^c = \left\{(w, x, z) \in \mathbb{R}^{1+n+n} : \right.$$

$$w \prod_{i \in S} z_i \geq \sum_{i \in S} \left( q_i x_i^2 \prod_{l \in S \setminus \{i\}} z_l \right) \quad S \subseteq \{1, 2, \ldots, n\}$$

$$u_i z_i \geq x_i \geq 0, \quad x_i \geq 0, \quad i = 1, 2, \ldots, n \right\} \quad (\Pi)$$

**Theorem**

$$\text{Proj}_{(w,x,z)}(\bar{Q}^c) = Q^c = \text{conv}(Q).$$
Descriptions in the Original Space

- We can also write a convex hull description in the original space of variables, by projecting out $y$:

$$Q^c = \left\{ (w, x, z) \in \mathbb{R}^{1+n+n} : \right.$$ 

$$w \prod_{i \in S} z_i \geq \sum_{i \in S} \left( q_i x_i^2 \prod_{l \in S \setminus \{i\}} z_l \right) S \subseteq \{1, 2, \ldots, n\}$$

$$u_i z_i \geq x_i \geq 0, \ x_i \geq 0, \ i = 1, 2, \ldots, n \right\} \ (\Pi) \n$$

**Theorem**

$$\text{Proj}_{(w,x,z)}(\tilde{Q}^c) = Q^c = \text{conv}(Q).$$

- $Q^c$ consists of an exponential number of nonlinear inequalities.
Extending the Intuition

- To deal with general convex sets, let $W = W^1 \cup W^0$:
  
  $$W^0 = \{(x, z) \in \mathbb{R}^{n+1} | x = 0, z = 0\}$$
  $$W^1 = \{(x, z) \in \mathbb{R}^{n+1} | f_k(x) \leq 0 \text{ for } k \in K, u \geq x \geq 0, z = 1\}$$

- Write an extended formulation (XF) for $\text{conv}(W)$

\[
\left\{(x, x_0, x_1, z, z_0, z_1, \alpha) \in \mathbb{R}^{3n+4} | 1 \geq \alpha \geq 0, x^0 = 0, z^0 = 0
\right.
\]
\[
x = \alpha x^1 + (1 - \alpha) x^0, z = \alpha z^1 + (1 - \alpha) z^0,
\]
\[
f_i(x^1) \leq 0 \text{ for } i \in I, u \geq x^1 \geq 0, z^1 = 1\]
Simplify, Simplify, Simplify

- Substitute out $x^0, z^0$ and $z^1$: They are fixed in $(XF)$
- $z = \alpha$ after these substitutions, so substitute it out as well.
- $x = \alpha x^1 = zx^1$, so we can eliminate $x^1$ by replacing it with $x/z$ provided that $z > 0$. 

---

Lemma

If $W_1$ is convex, then $\text{conv}(W) = W - \cup W_0$, where $W_n = \{(x, z) \in R^n_+ | f_k(x/z) \leq 0 \forall k \in K, uz \geq x \geq 0, 1 \geq z > 0\}$

Lemma Extension

$\text{conv}(W) = \text{closure}(W - \cup W_0)$
Simplify, Simplify, Simplify

- Substitute out $x^0, z^0$ and $z^1$: They are fixed in $(XF)$
- $z = \alpha$ after these substitutions, so substitute it out as well.
- $x = \alpha x^1 = zx^1$, so we can eliminate $x^1$ by replacing it with $x/z$ provided that $z > 0$.

Lemma

If $W^1$ is convex, then $\text{conv}(W) = W^- \cup W^0$, where

$$W^- = \left\{ (x, z) \in \mathbb{R}^{n+1} \mid f_k(x/z) \leq 0 \ \forall k \in K, uz \geq x \geq 0, 1 \geq z > 0 \right\}$$
Simplify, Simplify, Simplify

- Substitute out $x^0, z^0$ and $z^1$: They are fixed in $(XF)$
- $z = \alpha$ after these substitutions, so substitute it out as well.
- $x = \alpha x^1 = zx^1$, so we can eliminate $x^1$ by replacing it with $x/z$ provided that $z > 0$.

**Lemma**

If $W^1$ is convex, then $\text{conv}(W) = W^- \cup W^0$, where

$$W^- = \left\{ (x, z) \in \mathbb{R}^{n+1} \mid f_k(x/z) \leq 0 \ \forall k \in K, \ uz \geq x \geq 0, 1 \geq z > 0 \right\}$$

**Lemma Extension**

$$\text{conv}(W) = \text{closure}(W^-)$$
Convexify, Convexify, Convexify

- **Note:** $f_k(x/z)$ is not necessarily convex, even if $f_k(x)$ is.
- However, $zf_k(x/z)$ is convex if $f_k(x)$ is.
- Multiplying both sides of the inequality by $z > 0$ doesn’t change the set $W^-$:

  $$W^- = \left\{ (x, z) \in \mathbb{R}^{n+1} \mid zf_k(x/z) \leq 0 \ \forall k \in K, uz \geq x \geq 0, 1 \geq z > 0 \right\}$$

- You can, if you wish, multiply by $z^p$
Giving You Some Perspective

- For a convex function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, the function
  \[ P(f(z, x)) = z f(x/z) \]

  is known as the perspective function of $f$

- The epigraph of $P(f(z, x))$ is a cone pointed at the origin whose lower shape is $f(x)$
For a convex function $f(x) : \mathbb{R}^n \to \mathbb{R}$, the function

$$P(f(z, x)) = zf(x/z)$$

is known as the perspective function of $f$.

The epigraph of $P(f(z, x))$ is a cone pointed at the origin whose lower shape is $f(x)$.

Exploiting Your Perspective

- If $z_i$ is an indicator that the (nonlinear, convex) inequality $f(x) \leq 0$ must hold, (otherwise $x = 0$), replace the inequality with its perspective version:

$$z_i f(x/z_i) \leq 0$$

- The resulting (convex) inequality is a much tighter relaxation of the feasible region.
An Axioma Connection

**Stubbs (1996)**

- In his Ph.D. thesis, Stubbs gives (without proof) $\text{conv}(\bar{Q})$, our original (high-dimensional) set.
An Axioma Connection

**Stubbs (1996)**
- In his Ph.D. thesis, Stubbs gives (without proof) $\text{conv}(\bar{Q})$, our original (high-dimensional) set.

**Ceria and Soares (1999)**
- Describe $K = \bigcup_{i \in M} K_i$, with $K_i = \{x \mid f_i(x) \leq 0\}$ in a higher-dimensional space.
- $x \in \text{conv}(K) \iff$

$$x = \sum_{i \in M} \lambda_i x_i, \ P(f_i(\lambda_i, x_i)) \leq 0, \lambda \in \Delta_{|M|}$$
Other Smart People

Frangioni and Gentile (2006)

- Study: $y \geq f(x), \ x \leq uz$, give **perspective cut**:
  
  $$y \geq f(x) + \nabla f(x)^T(x - \hat{x}) - (\hat{x}^T \nabla f(\hat{x}) + f(\hat{x}))(z - 1)$$

- This is first-order Taylor expansion of perspective $zf(x/z) + y \leq 0$ about $(\hat{x}, f(\hat{x}), 1)$

- Feasible inequality by convexity of $f(x)$
Other Smart People

Frangioni and Gentile (2006)

- Study: \( y \geq f(x), x \leq uz \), give perspective cut:
  \[
  y \geq f(x) + \nabla f(x)^T(x - \hat{x}) - (\hat{x}^T \nabla f(\hat{x}) + f(\hat{x}))(z - 1)
  \]
- This is first-order Taylor expansion of perspective \( zf(x/z) + y \leq 0 \) about \((\hat{x}, f(\hat{x}), 1)\)
- Feasible inequality by convexity of \( f(x) \)

Aktürk, Atamtürk, and Gürel (2007)

- Apply perspective reformulation (of epigraph indicator MINLP) to nonlinear machine scheduling problem
- Explain that formulations are representable as SOCP.
Facility Location

- $M$: Facilities
- $N$: Customers
- $x_{ij}$: percentage of customer $i$’s demand served from facility $j$
- $z_i = 1 \iff$ facility $i$ is opened
- Fixed cost for opening facility $i$
- **Quadratic** cost for serving $j$ from $i$
- Problem studied by Günlük, Lee, and Weismantel (’07), and classes of strong cutting planes derived
Separable Quadratic UFL—Formulation

\[ z^* \overset{\text{def}}{=} \min \sum_{i \in M} c_i z_i + \sum_{i \in M} \sum_{j \in N} q_{ij} x_{ij}^2 \]

subject to

\[ x_{ij} \leq z_i \quad \forall i \in M, \forall j \in N \]
\[ \sum_{i \in M} x_{ij} = 1 \quad \forall j \in N \]
\[ x_{ij} \geq 0 \quad \forall i \in M, \forall j \in N \]
\[ z_i \in \{0, 1\} \quad \forall i \in M \]
Separable Quadratic UFL—Formulation

\[ z^* \overset{\text{def}}{=} \min \sum_{i \in M} c_i z_i + \sum_{i \in M} \sum_{j \in N} q_{ij} y_{ij} \]

subject to

\[ x_{ij} \leq z_i \quad \forall i \in M, \forall j \in N \]
\[ \sum_{i \in M} x_{ij} = 1 \quad \forall j \in N \]
\[ x_{ij} \geq 0 \quad \forall i \in M, \forall j \in N \]
\[ z_i \in \{0, 1\} \quad \forall i \in M \]
\[ x_{ij}^2 - y_{ij} \leq 0 \quad \forall i \in M, \forall j \in N \]
Separable Quadratic UFL—Formulation

\[ z^* \overset{\text{def}}{=} \min \sum_{i \in M} c_iz_i + \sum_{i \in M} \sum_{j \in N} q_{ij}y_{ij} \]

subject to

\[ x_{ij} \leq z_i \quad \forall i \in M, \forall j \in N \]
\[ \sum_{i \in M} x_{ij} = 1 \quad \forall j \in N \]
\[ x_{ij} \geq 0 \quad \forall i \in M, \forall j \in N \]
\[ z_i \in \{0, 1\} \quad \forall i \in M \]
\[ x_{ij}^2 - z_iy_{ij} \leq 0 \quad \forall i \in M, \forall j \in N \]
Strength of Relaxations

- $z_R$: Value of NLP relaxation
- $z_{GLW}$: Value of NLP relaxation after GLW cuts
- $z_P$: Value of perspective relaxation
- $z^*$: Optimal solution value
## Strength of Relaxations

- $z_R$: Value of NLP relaxation
- $z_{GLW}$: Value of NLP relaxation after GLW cuts
- $z_P$: Value of perspective relaxation
- $z^*$: Optimal solution value

| $|M|$ | $N$ | $z_R$ | $z_{GLW}$ | $z_P$ | $z^*$ |
|-----|-----|-------|-----------|-------|-------|
| 10  | 30  | 140.6 | 326.4     |       | 348.7 |
| 15  | 50  | 141.3 | 312.2     |       | 384.1 |
| 20  | 65  | 122.5 | 248.7     |       | 289.3 |
| 25  | 80  | 121.3 | 260.1     |       | 315.8 |
| 30  | 100 | 128.0 | 327.0     |       | 393.2 |
## Strength of Relaxations

- $z_R$: Value of NLP relaxation
- $z_{GLW}$: Value of NLP relaxation after GLW cuts
- $z_P$: Value of perspective relaxation
- $z^*$: Optimal solution value

<table>
<thead>
<tr>
<th>M</th>
<th>N</th>
<th>$z_R$</th>
<th>$z_{GLW}$</th>
<th>$z_P$</th>
<th>$z^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>30</td>
<td>140.6</td>
<td>326.4</td>
<td>346.5</td>
<td>348.7</td>
</tr>
<tr>
<td>15</td>
<td>50</td>
<td>141.3</td>
<td>312.2</td>
<td>380.0</td>
<td>384.1</td>
</tr>
<tr>
<td>20</td>
<td>65</td>
<td>122.5</td>
<td>248.7</td>
<td>288.9</td>
<td>289.3</td>
</tr>
<tr>
<td>25</td>
<td>80</td>
<td>121.3</td>
<td>260.1</td>
<td>314.8</td>
<td>315.8</td>
</tr>
<tr>
<td>30</td>
<td>100</td>
<td>128.0</td>
<td>327.0</td>
<td>391.7</td>
<td>393.2</td>
</tr>
</tbody>
</table>
Design of Uncongested Network

- Capacitated directed network: $G = (N, A)$
- Set of commodities: $K$
- Node demands: $b^k_i$
  \[ \forall i \in N, \forall k \in K \]
- Each arc $(i, j) \in A$ has
  - Fixed cost: $c_{ij}$
  - Capacity: $u_{ij}$
  - Queueing weight: $r_{ij}$
Design of Uncongested Network

- Capacitated directed network: \( G = (N, A) \)
- Set of commodities: \( K \)
- Node demands: \( b^k_i \)
  \[ \forall i \in N, \forall k \in K \]
- Each arc \((i, j) \in A\) has
  - Fixed cost: \( c_{ij} \)
  - Capacity: \( u_{ij} \)
  - Queueing weight: \( r_{ij} \)

- \( z_{ij} \in \{0, 1\} \): Indicates whether arc \((i, j) \in A\) is opened.
- \( x^k_{ij} \): The quantity of commodity \( k \) routed on arc \((i, j)\)
Network Design

- Let $f_{ij} \overset{\text{def}}{=} \sum_{k \in K} x_{ij}^k$ be the flow on arc $(i,j)$.

- A measure of queueing delay is:

\[
\rho(f) \overset{\text{def}}{=} \sum_{(i,j) \in A} r_{ij} \frac{f_{ij}}{1 - f_{ij}/u_{ij}}
\]

\[
\frac{f}{(1 - f/u)}
\]

\[
f = u
\]
Network Design

Let $f_{ij} \overset{\text{def}}{=} \sum_{k \in K} x_{ij}^k$ be the flow on arc $(i, j)$.

A measure of queueing delay is:

$$\rho(f) \overset{\text{def}}{=} \sum_{(i,j) \in A} r_{ij} \frac{f_{ij}}{1 - f_{ij}/u_{ij}}$$

f/(1 - f/u)

Our Network Design Problem

Design network to keep total queueing delay less than a given value $\beta$, and this is to be accomplished at minimum cost.
Network Design Formulation

\[
\begin{align*}
\min & \quad \sum_{(i,j) \in A} c_{ij} z_{ij} \\
\text{s.t.} & \quad \sum_{(j,i) \in A} x_{ij}^k - \sum_{(i,j) \in A} x_{ij}^k = b_i^k \quad \forall i \in N, \forall k \in K \\
& \quad \sum_{k \in K} x_{ij}^k - f_{ij} = 0 \quad \forall (i,j) \in A \\
& \quad f_{ij} \leq u_{ij} z_{ij} \quad \forall (i,j) \in A \\
& \quad y_{ij} \geq \frac{r_{ij} f_{ij}}{1 - f_{ij}/u_{ij}} \quad \forall (i,j) \in A \\
& \quad \sum_{(i,j) \in A} y_{ij} \leq \beta
\end{align*}
\]
Perspective Formulations and Cones

Consider the nonlinear inequality:

\[ y \geq \frac{rf}{1 - f/u} \iff ruf \leq y(u - f) \]
Perspective Formulations and Cones

- Consider the nonlinear inequality:

\[ y \geq \frac{rf}{1 - f/u} \iff ruf \leq y(u - f) \]

- Since \( z_{ij} = 0 \Rightarrow f_{ij} = 0 \), we can write the perspective reformulation:

\[ \frac{y}{z} \geq \frac{rf}{1 - f/zu} \iff ruzf \leq y(uz - f) \]
Perspective Formulations and Cones

- Consider the nonlinear inequality:
  \[ y \geq \frac{rf}{1 - f/u} \iff ruf \leq y(u - f) \]

- Since \( z_{ij} = 0 \Rightarrow f_{ij} = 0 \), we can write the perspective reformulation:
  \[ y/z \geq \frac{rf/z}{1 - f/zu} \iffruzf \leq y(uz - f) \]

Cones Are Everywhere!

- The inequalities \( ruf \leq y(u - f) \) and \( urfz \leq y(uz - f) \) are SOC-representable:
  \[ ruf \leq y(u - f) \iff rf^2 \leq (y - rf)(u - f) \]
  \[ rufz \leq y(uz - f) \iff rf^2 \leq (y - rf)(uz - f) \]

since \( y \geq rf, \ u \geq f, \ uz \geq f \)
Results (Under Construction)

- ZIB SNDLIB instance: ATL.
- $|N| = |K| = 15$, $|A| = 22$
- Instance solved using (beta) version of Mosek (v5) conic MIP solver
- No fancy cutting planes (cut-set inequalities) added
Results (Under Construction)

- ZIB SNDLIB instance: ATL.
- $|N| = |K| = 15$, $|A| = 22$
- Instance solved using (beta) version of Mosek (v5) conic MIP solver
- No fancy cutting planes (cut-set inequalities) added

<table>
<thead>
<tr>
<th></th>
<th>Nodes</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Perspective</td>
<td>3686</td>
<td>517.1</td>
</tr>
<tr>
<td>W/Perspective</td>
<td>414</td>
<td>52.5</td>
</tr>
</tbody>
</table>

ATL Network

Günlük and Linderoth (UW-Madison) A Different Perspective on Perspective Cuts
Conclusions

Jeff Linderoth Gives Really Stupid Talks
Jeff Linderoth Gives Really Stupid Talks
Jeff Linderoth Gives Really Stupid Talks
Jeff Linderoth Gives Really Stupid Talks
Jeff Linderoth Gives Really Stupid Talks
Jeff Linderoth Gives Really Stupid Talks
Other Conclusions

- Strong reformulations for MINLPs are likely to be just as important as they are for MILPs.
- Strong formulations for MINLPs may require nonlinear inequalities. (Duh!)
- Much of the work we present here has (recently) found its way into the literature.
Other Conclusions

- Strong reformulations for MINLPs are likely to be just as important as they are for MILPs.
- Strong formulations for MINLPs may require nonlinear inequalities. (Duh!)
- Much of the work we present here has (recently) found its way into the literature.

Our “contributions”

- Give convex hull for the union of a (general) bounded convex set and a point.
- Give description in original space of variables.
- Exploit SOC-representability of strong reformulations to solve instances much more effectively.