A Gentle Introduction to Stochastic Programming

Jeff Linderoth

Department of Industrial and Systems Engineering
Lehigh University

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Outline

- What is Stochastic Programming (SP)?
  - There are lots of stochastic programming problems
  - The Canonical Problem
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- **What is Stochastic Programming (SP)?**
  - There are *lots* of stochastic programming problems
  - The **Canonical Problem**

- **Solving Stochastic Programs**
  - Deterministic equivalents
  - Sampling
  - A decomposition algorithm
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- What is Stochastic Programming (SP)?
  - There are *lots* of stochastic programming problems
  - The **Canonical Problem**
- Solving Stochastic Programs
  - Deterministic equivalents
  - Sampling
  - A decomposition algorithm
- Stochastic Integer Programming
  - It’s Very Hard
Why Care about Stochastic Programming?

What we anticipate seldom occurs; what we least expected generally happens.

Benjamin Disraeli (1804 - 1881)
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▶ Think of Stochastic Programming (SP) as Mathematical Programming (MP) with random parameters
▶ This is useful, since we often really don’t know the data
  ▶ Customer demands
  ▶ Market actions
  ▶ Insert your own favorite uncertainty here...
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- Think of Stochastic Programming (SP) as Mathematical Programming (MP) with random parameters
- This is useful, since we often really don’t know the data
  - Customer demands
  - Market actions
  - Insert your own favorite uncertainty here...
- SP assumes a probability distribution for the random variable $(\omega)$ is known or can be approximated with reasonable accuracy
Mathematical Formulations

A Mathematical Program

\[ \min_{x \in X} f(x) \quad (\text{MP}) \]

\[ X \overset{\text{def}}{=} \{ x \in X_0 \mid g_i(x) \leq 0 \quad \forall i \in M \} \]
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A Stochastic Program

\[
\min_{x \in X(\omega)} F(x, \omega) \quad (\text{SP})
\]
But I Haven’t Told you Anything!

How should we deal with the randomness?
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How should we deal with the randomness

\[
\text{min } F(x, \hat{\omega}) \quad \text{Point Estimate}
\]
But I Haven’t Told you Anything!

How should we deal with the randomness

- \( \min F(x, \hat{\omega}) \)  
  - Point Estimate
- \( \min \mathbb{E}_\omega F(x, \omega) \)  
  - Risk Neutral
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- $\min \mathbb{E}_\omega F(x, \omega)$  
  Risk Neutral
- $\min \mathbb{E}_\omega F(x, \omega) - \lambda \rho(F(x, \omega))$
  Risk Measures
  - $\rho(F(x, \omega)) = \text{Var}F(x, \omega)$  
  Markowitz
But I Haven’t Told you Anything!

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  - $\rho(F(x, \omega)) = \mathbb{E} [(F(x, \omega) - \mathbb{E} F(x, \omega))^+]$
  **Risk Measures**

  - Markowitz
  - Semideviation
Coping with Randomness—The Constraints

- \( X(\omega) = \{ x \in X_0 \mid G_i(x, \hat{\omega}) \leq 0 \quad \forall i \in M \} \)
- Point Estimate
Coping with Randomness—The Constraints

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  - For all realizations
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- \( X(\omega) = \{ x \in X_0 \mid G_i(x, \omega) \leq 0 \quad \forall \omega \in \mathcal{U} \subset \Omega \} \)
  - Robust Optimization
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- \( X(\omega) = \{ x \in X_0 \mid \mathbb{P}\{G_i(x, \omega) \leq 0 \, \forall i \in M \} \geq 1 - \alpha \} \)
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  - Individual Chance Constraints
Things People Want

Continuous Distributions
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(Conditional) Value at Risk
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- Continuous Distributions
- Network Problems
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Why Do I Care?
Different Strokes for Different Folks
THE Stochastic Program—Recourse Problems
Things People Want

- Continuous Distributions
- (Conditional) Value at Risk
- Scenario Trees
- Network Problems
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- (Joint) Chance Constraints
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- Free Beer

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A Gentle Introduction to Stochastic Programming
What is Stochastic Programming
Stochastic Linear Programming
Stochastic Integer Programming

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Supporting Stochastic Programs

- I point out all these different flavors of SP to highlight what I think has been one of the hinderances of acceptance of stochastic programming in the broader community.
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- **An Anecdote.** ISMP XVIII, Copenhagen, 2003.
  - Irv Lustig, “Optimization Envangelist”, ILOG
The Canonical Problem—Multistage Linear Recourse

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  - I think this is very likely the most useful “stochastic program”.

Typically, we must make decision $x$ before $\omega$ is known. But we have some recourse once we know $\omega$.

1. We make a decision now (first-period decision)
2. Nature makes a random decision (“stuff” happens)
3. We make a second period decision that attempts to repair the havoc wrought by nature in (2). (recourse)
The Canonical Problem—Multistage Linear Recourse

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Let’s do a simple model...
Random Linear Programming

Everyone’s Favorite Problem. The Linear Program.

\[
\min_{x \in X} \{ c^T x \mid Ax = b \}
\]

\[X = \{ x \in \mathbb{R}^n : l \leq x \leq u \}\]
Random Linear Programming

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\min_{x \in X} \{ c^T x \mid Ax = b \}
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- \( X = \{ x \in \mathbb{R}^n : l \leq x \leq u \} \)

- What if some parameters are random?

\[
\min_{x \in X} \{ c^T x \mid Ax = b, T(\omega)x = h(\omega) \}
\]
The Recourse Game

- Again, we are interested in solving decision problems where the decision $x$ must be made before the realization of $\omega$ is known.
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- In recourse models, the random constraints are modeled as “soft” constraints. Possible violation is accepted, but the cost of violations will influence the choice of \( x \).

- In fact, a second-stage linear program is introduced that will describe how the violated random constraints are dealt with.
Penalizing Shortfall with $LP(\omega)$

In the simplest case, we may just count penalize deviation in the constraints by penalty coefficient vectors $q_+$ and $q_-$

minimize

$$c^T x + q_+^T s(\omega) + q_-^T t(\omega)$$

subject to

$$Ax = b$$

$$T(\omega)x + s(\omega) - t(\omega) = h(\omega)$$

$$x \in X$$
The New Optimization Problem

- So then, a reasonable problem to solve (to deal with the randomness) is...

minimize

\[ c^T x + \mathbb{E}_\omega [q^T_+ s(\omega) + q^T_- t(\omega)] \]

subject to

\[ Ax = b \]

\[ T(\omega)x + s(\omega) - t(\omega) = h(\omega) \quad \forall \omega \in \Omega \]

\[ x \in X \]
Recourse

- In general, we can *react* in an intelligent (or optimal) way.
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    - \( Y = \{y \in \mathbb{R}^p : y \geq 0\} \)
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- A set $Y \in \mathbb{R}^p$ that describes the feasible set of recourse actions.
  - $Y = \{y \in \mathbb{R}^p : y \geq 0\}$
- $q$: a vector of recourse costs.
Recourse

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    - $Y = \{y \in \mathbb{R}^p : y \geq 0\}$
  - $q$: a vector of recourse costs.
  - $W$: a $m \times p$ matrix, called the *recourse matrix*
A Recourse Formulation

minimize

\[ c^T x + \mathbb{E}_\omega [q^T y] \]

subject to

\[ Ax = b \]
\[ T(\omega)x + Wy(\omega) = h(\omega) \quad \forall \omega \in \Omega \]
\[ x \in X \]
\[ y(\omega) \in Y \]
Writing With the y’s

\[ \min_{x \in \mathbb{R}^n, y(\omega) \in \mathbb{R}^p} \mathbb{E}_\omega \left[ c^T x + q^T y(\omega) \right] \]

subject to

\[ Ax = b \quad \text{First Stage Constraints} \]

\[ T(\omega)x + W y(\omega) = h(\omega) \quad \forall \omega \in \Omega \quad \text{Second Stage Constraints} \]

\[ x \in X \quad y(\omega) \in Y \]

- Imagine the case where \( \Omega = \{\omega_1, \omega_2, \ldots \omega_S\} \subseteq \mathbb{R}^r \).
- \( P(\omega = \omega_s) = p_s, \forall s = 1, 2, \ldots, S \)
- \( T_s \equiv T(\omega_s), h_s = h(\omega_s) \)
Deterministic Equivalent

We can then write the deterministic equivalent as:

\[ \begin{align*}
    c^T x &+ p_1 q^T y_1 + p_2 q^T y_2 + \cdots + p_s q^T y_s \\
\text{s.t.} & \\
    Ax &= b \\
    T_1 x + W y_1 &= h_1 \\
    T_2 x + W y_2 &= h_2 \\
    \vdots & \quad + \quad \vdots \\
    T_s x + W y_s &= h_s \\
    x \in X & \quad y_1 \in Y \quad y_2 \in Y \quad \cdots \quad y_s \in Y
\end{align*} \]
About the DE

- $y_s \equiv y(\omega_s)$ is the recourse action to take if scenario $\omega_s$ occurs.
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- Imagine the following (real) problem. A Telecom company wants to expand its network in a way in which to meet an unknown (random) demand.

There are 86 unknown demands. Each demand is independent and may take on one of five values.

\[
S = |\Omega| = \prod_{k=1}^{86} (5) = 5^{86} = 4.77 \times 10^{72}
\]

The number of subatomic particles in the universe.
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    - The number of subatomic particles in the universe.
Why Don’t More People Use Stochastic Programming
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They don’t start their training early enough!
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Why Don’t More People Use Stochastic Programming

- Because they can’t “solve” them? **Try Sampling!**
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\[
\min_{x \in X} \{ f(x) \overset{\text{def}}{=} \mathbb{E}_P F(x, \omega) \equiv \int_\Omega F(x, \omega) dP(\omega) \}
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\]

- Draw \( \omega^1, \omega^2, \ldots, \omega^N \) from \( P \)
- Sample Average Approximation (SAA):

\[
\hat{f}_N(x) \equiv N^{-1} \sum_{j=1}^{N} g(x, \omega^j)
\]
Why Don’t More People Use Stochastic Programming

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▶ Draw \( \omega^1, \omega^2, \ldots, \omega^N \) from \( P \)

▶ Sample Average Approximation (SAA):

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▶ \( \hat{f}_N(x) \) is an unbiased estimator of \( f(x) \) (\( \mathbb{E}[\hat{f}_N(x)] = f(x) \)).
Why Don’t More People Use Stochastic Programming

- Because they can’t “solve” them? Try Sampling!

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- \( \hat{f}_N(x) \) is an unbiased estimator of \( f(x) \) (\( \mathbb{E}[\hat{f}_N(x)] = f(x) \)).

- Minimize the SAA: \( \min_{x \in X} \{ \hat{f}_N(x) \} \)
Sampling is Good!

- For two-stage stochastic recourse problems, some *very interesting* recent theory of Shapiro and Homem-de-Mello has shown that you need *shockingly few scenarios* in order for the solution of the sample average approximation to be a very good solution to the *true* problem.

- This theory has been backed up with computational experience.
  - For a problem with $10^{81}$ scenarios, a 100 scenario sample was sufficient.
  - A different instance with $10^{70}$ scenarios required around a 5000 scenario sample.
Solving “Medium Sized” Problems

- Formulate as “two-level” problem

\[
\begin{align*}
\min_{x \in \mathbb{R}^n_+ : Ax = b} & \left\{ c^T x + \mathbb{E}_\omega \left[ \min_{y \in Y} \{ q^T y : W y = h(\omega) - T(\omega)x \} \right] \right\} \\
\end{align*}
\]

- Second stage value function, or Cost-to-go function

\[ v : \mathbb{R}^m \mapsto \mathbb{R}. \]

\[ v(z) \equiv \min_{y \in Y} \{ q^T y : W y = z \}, \]

- Expected Value Function, or Expected cost-to-go function

\[ Q : \mathbb{R}^n \mapsto \mathbb{R}. \]

\[ Q(x) \equiv \mathbb{E}_\omega [v(h(\omega) - T(\omega)x)] \]

- For any policy \( x \in \mathbb{R}^n \), it describes the expected cost of the recourse.
The SP Problem

- Using these definitions, we can write our recourse problem in terms only of the $x$ variables:

$$\min_{x \in X} \{ c^T x + Q(x) : Ax = b \}$$

- This is a (nonlinear) programming problem in $\mathbb{R}^n$.

- The ease of solving such a problem depends on the properties of $Q(x)$.

- $Q(x)$ is...
  - Convex...
  - Continuous...
  - Non-Differentiable
Important (and well-known) Facts

- $Q(x)$ is a piecewise linear convex function of $x$.
- If $\pi_i$ is an optimal dual solution to the linear program corresponding to $i$th scenario, then $T_i^T \pi_i \in \partial Q(\hat{x})$
- $g(\hat{x}) \overset{\text{def}}{=} \sum_{i \in S} p_i T_i^T \pi_i \in \partial Q(\hat{x})$.
- Evaluation of $Q(\hat{x})$ is separable
  - We can solve linear programs corresponding to each $Q(\hat{x})$ independently – in parallel!
Best-Known Solution Procedure
Best-Known Solution Procedure
L-shaped method

- Represent $Q(x)$ by an artificial variable $\theta$ and find supporting planes for $\theta$ (from subgradients of $Q(x^k)$).
  - $\theta \geq g(x^k)^T x + (Q(x^k) - g^T x^k)$ (*)
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1. Solve the **master problem** with the current $\theta_j$-approximations to $Q(x)$ for $x^k$. 

$\theta_j$-approximations for $Q(x)$
L-shaped method

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1. Solve the **master problem** with the current $\theta_j$-approximations to $Q(x)$ for $x^k$.
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3. $k = k+1$. Goto 1.
Worth 1000 Words

$Q(x)$

$x$
Worth 1000 Words

\[ Q(x) \]

\[ x \]

\[ x^k \]
Worth 1000 Words

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Even Harder—Multistage Problems

- Multistage problems are defined by a sequence of decision, event, decision, event, . . . , decision.
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- Multistage problems are defined by a sequence of decision, event, decision, event, . . . , decision.
- Multistage problems are even bigger (scenarios grow again at a rate exponential in the number of stages)
Even Harder—Multistage Problems

- Multistage problems are defined by a sequence of decision, event, decision, event, . . . , decision.
- Multistage problems are even bigger (scenarios grow again at a rate exponential in the number of stages)
- We have to keep track of the random event “structure”—the scenario tree—and its relationship to the decisions that we make
Existing(?) Tools

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▶ I am happy to show off the XPRESS-SP tool if anyone is interested.
Stochastic MIP

- Recall that if $\Omega$ was finite, we could write the (deterministic equivalent) of a stochastic LP
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- Just a large scale LP

We can do the same for stochastic MIP

- Just a large-scale IP
- But a large-scale IP with a very weak linear programming relaxation $\Rightarrow$ not likely to be solved by “off-the-shelf” software like cplex.
Nasty, Nasty, Functions

- If you didn’t fall asleep during the mathy part, recall that our L-Shaped method for stochastic LP was based on knowing “nice” properties of the second stage value function \( v(z) \) or the Expected Value Function \( Q(x) \).

- For IP...

\[
v(z) = \min_{y \in \mathbb{Z}^n_+} \{ q^T y | Wy = z \}
\]

- Here are two properties...
  - \( v(z) \) is lower semicontinuous on \( \mathbb{R}^m \)
  - The discontinuity points of \( v \) are contained in a countable union of hyperplanes in \( \mathbb{R}^m \)

- These are not very powerful properties
Algorithms for Stochastic IP

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- Structured Enumeration
  - Based on strange mathematical entities like test sets and Groebner Bases
Conclusions

- You cannot condense stochastic programming into a one-hour course
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- Jeff should not wait until the last minute to prepare his slides
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The Major Conclusion

Stochastic Programming is worthwhile to study a bit more!
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- Stochastic Integer Programming is going to be our next topic
- Suvrajeet Sen from NSF will come speak in Friday Seminar on 9/17
- We’re going to read a survey paper for next week.

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