Un Marco de CVaR para Optimización con Intereses Múltiples

(A CVaR Framework for Multi-Stakeholder Optimization)

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Multiobjective Optimization

\[ \min_{x \in \mathcal{X}} (f_1(x), f_2(x), \ldots, f_n(x)) \]

**Utopia Point**

\[ f_i := \min_{x \in \mathcal{X}} f_i(x), \quad i \in \mathcal{O} := \{1..n\} \]

\[ x_i := \arg \min_{x \in \mathcal{X}} f_i(x), \quad i \in \mathcal{O} \]

**Nadir Point**

\[ \overline{f}_i := \max\{f_i(x_1), f_i(x_2), \ldots, f_i(x_n)\}, \quad i \in \mathcal{O} \]

**Re-Scaling**

\[ \hat{f}_i(x) \leftarrow \frac{f_i(x) - f_i}{\overline{f}_i - f_i}, \quad i \in \mathcal{O} \]

**Compromise Solution**

\[ x^* = \min_{x \in \mathcal{X}} \|f(x)\|_p \]

\[ f(x) := (f_1(x), f_2(x), \ldots, f_n(x)) \]

**Issues:**

- **Ambiguity:** Meaning of Compromise?
- **Dimensionality:** Construct Pareto Set?
Definition: (Weak Pareto Optimality) A decision $x^*$ with $f_i(x^*), i \in \mathcal{O}$ is a weakly Pareto optimal solution of MOO if there does not exist an alternative solution $\bar{x}$ with objectives $f_i(\bar{x}), i \in \mathcal{O}$ satisfying $f_i(\bar{x}) < f_i(x^*)$ for all $i \in \mathcal{O}$.

Definition: (Pareto Optimality) A decision $x^*$ with $f_i(x^*), i \in \mathcal{O}$ is a Pareto optimal solution of MOO if there does not exist an alternative solution $\bar{x}$ with objectives $f_i(\bar{x}), i \in \mathcal{O}$ satisfying $f_i(\bar{x}) \leq f_i(x^*)$ for all $i \in \mathcal{O}$ and at least one index $i$ satisfying $f_i(\bar{x}) < f_i(x^*)$. 
**Toma de Decisiones con Intereses Múltiples – Multistakeholder Optimization**

**Ideal Stakeholder Solution**

\[
x_j^* := \arg\min_{x \in \mathcal{X}} w_j^T f(x), \quad j \in \mathcal{S} := \{1..m\}
\]

\[w_j : \text{Stakeholder Priority Vector}\]

**Stakeholder Dissatisfaction Function**

\[
d_j(x) := w_j^T (f(x) - f_j^*)
\]

\[= w_j^T f(x) - w_j^T f_j^*\]

\[f_j^* := f(x_j^*)\]

**Average Dissatisfaction** *Dyer, 1992*

\[
\min_{x \in \mathcal{X}} \frac{1}{m} \sum_{j \in \mathcal{S}} d_j(x)
\]

**Worst-Case Dissatisfaction** *Mehrotra, 2012*

\[
\min \max_{x \in \mathcal{X}} \{d_j(x)\}
\]

**Conditional Value-at-Risk** *This Work*

\[
\min_{x \in \mathcal{X}} \text{CVaR}_\alpha(d(x))
\]

**Key Observation:** Interpret Opinions as Samples from Population
Key Property:

\[
\operatorname{CVaR}_\alpha [d(x)] = \min \nu \frac{1}{m} \sum_{j=1}^{m} \left[ \nu + \frac{1}{1 - \sigma} [d_j(x) - \nu]_+ \right]
\]

\[
\lim_{\alpha \to 0} \operatorname{CVaR}_\alpha [d(x)] = \frac{1}{m} \sum_{j \in S} d_j(x)
\]

\[
\lim_{\alpha \to 1} \operatorname{CVaR}_\alpha [d(x)] = \max_{j \in S} \{ d_j(x) \}
\]

Question: Are CVaR Solutions Pareto Optimal?
Interpretación Geométrica - Geometric Interpretation

Disagreement Vector

\[ d_j(x) := w_j^T (f_j(x) - f_j^*), \quad j \in S \]

\[ d(x) := [d_1(x), d_2(x), \ldots, d_m(x)] \]

**Definition: (Scaled L_p norm).** Consider a fixed decision \( x \in \mathbb{R}^{n_x} \) and the dissatisfaction vector \( d(x) \in \mathbb{R}^m \). The scaled \( L_p \) norm (denoted as \( L_p^m \)) of \( d(x) \) is defined as,

\[
\|d(x)\|_p^m := \left( \frac{1}{m} \sum_{j=1}^{m} |d_j(x)|^p \right)^{\frac{1}{p}}, \quad p \geq 1.
\]

The scaled \( L_p \) norm has the following extreme cases,

\[
\|d(x)\|_1^m = \frac{1}{m} \sum_{j=1}^{m} |d_j(x)|
\]

\[
\|d(x)\|_{\infty}^m = \max_j |d_j(x)|.
\]
Interpretación Geométrica - Geometric Interpretation

\[
\begin{align*}
\min_{x \in \mathcal{X}} \ & \frac{1}{m} \sum_{j \in \mathcal{S}} d_j(x) \\
\min \max_{x \in \mathcal{X}, j \in \mathcal{S}} \{d_j(x)\} \\
\min_{x \in \mathcal{X}} \ & \text{CVaR}_\alpha(d(x)) \\
\min_{x \in \mathcal{X}} \ & \|d(x)\|_1^m \\
\min_{x \in \mathcal{X}} \ & \|d(x)\|_\infty^m
\end{align*}
\]
Definition: (Scaled CVaR norm). Consider the vector $d(x) \in \mathbb{R}^m$ and assume (without loss of generality) that $d_1(x) \leq d_2(x) \leq \cdots \leq d_m(x)$ holds. Define also the scalars $\alpha_j := \frac{j}{m}$, $j = 0, ..., m-1$. The scaled CVaR norm of vector $d(x)$ with parameter $\alpha_j$ is defined as,

$$
\ll d(x) \rr_{\alpha_j}^m := \frac{1}{m-j} \sum_{i=j+1}^{m} d_i(x).
$$

CVaR Norm Properties: For fixed $x$ consider the discrete random variable $d(x)$ with outcomes $d_1(x), d_2(x), ..., d_m(x)$, probabilities $p_j = \frac{1}{m}$, $j \in S$, and the corresponding vector $d(x)$.

i) $\ll \cdot \rr_{\alpha}^m$ is a Norm for $\alpha \in [0, 1]$

ii) $\ll d(x) \rr_{\alpha}^m = CVaR_{\alpha}(d(x))$ for $\alpha \in [0, 1]$.

iii) $\ll d(x) \rr_{0}^m = \|d(x)\|_1^m$

iv) $\ll d(x) \rr_{\alpha}^m = \|d(x)\|_{\infty}^m$ for $\frac{m-1}{m} \leq \alpha \leq 1$.

v) For $\alpha$ such that $\alpha_j < \alpha < \alpha_{j+1}$, $j = 0, ..., m-2$:

$$
\ll d(x) \rr_{\alpha}^m = \mu \ll d(x) \rr_{\alpha_j}^m + (1 - \mu) \ll d(x) \rr_{\alpha_{j+1}}^m
$$

with $\mu := \frac{(\alpha_{j+1} - \alpha)(1 - \alpha_j)}{(\alpha_{j+1} - \alpha_j)(1 - \alpha)}$.

vi) $\ll d(x) \rr_{\alpha}^m$ is a nondecreasing function of $\alpha \in [0, 1]$.

Norm Conditions:

Homogeneity:

$$
\rho(\lambda x) = \lambda \rho(x)
$$

Subadditivity:

$$
\rho(x_1, x_2) \leq \rho(x_1) + \rho(x_2)
$$

Normalized:

$$
\rho(0) = 0
$$
La Norma CVaR - The CVaR Norm *Pavlikov & Uryasev, 2014*

**CVaR Norm**

\[
\begin{align*}
\min_{x \in \mathcal{X}} \; \text{CVaR}_\alpha [d(x)] & \iff 
\min_{x \in \mathcal{X}} \; \left\langle d(x) \right\rangle_\alpha^m & \iff 
\min_{(x,y) \in \mathcal{X} \times \mathbb{R}} \; y + \frac{1}{(1-\alpha)m} \sum_{j=1}^{m} (d_j(x) - y)_+
\end{align*}
\]

**LS Norm**

CVaR Norm Combinatorial But Can be Computed Using Continuous Formulation
Optimalidad de Pareto de las Soluciones de CVaR (Pareto Optimality of CVaR Solutions)

MOO
\[
\min_{x \in \mathcal{X}} (f_1(x), f_2(x), \ldots, f_n(x))
\]

CVaR Problem
\[
\min_{x \in \mathcal{X}} \ll d(x) \gg_{\alpha}^m \iff \min_{(x, y) \in \mathcal{X} \times \mathbb{R}} y + \frac{1}{(1 - \alpha)m} \sum_{j=1}^{m} (d_j(x) - y) +
\]

**Lemma:** Consider decisions \(\bar{x}, x^*\) with corresponding \(d_j(\bar{x}), d_j(x^*)\). We have:
\[
d_j(\bar{x}) < d_j(x^*), \quad j \in \mathcal{S} \quad \implies \quad \langle d(\bar{x}) \rangle_{\alpha}^m < \langle d(x^*) \rangle_{\alpha}^m, \quad \alpha \in [0, 1].
\]

**Theorem:** Let \(x^*\) be a solution of the CVaR problem. We have:

1. If \(w_j^{(i)} \geq 0, \quad j \in \mathcal{S}, \quad i \in \mathcal{O}\) then \(x^*\) is weak Pareto for MOO \(\forall \alpha \in [0, 1]\).

2. If \(w_j^{(i)} > 0, \quad j \in \mathcal{S}, \quad i \in \mathcal{O}\) then \(x^*\) is Pareto for MOO \(\forall \alpha \in [0, 1]\).

**Proof of Weak Pareto:** \(x^*\) is optimal for CVaR and thus \(\langle d(x) \rangle_{\alpha}^m \geq \langle d(x^*) \rangle_{\alpha}^m\) for any \(x \in \mathcal{X}\). Assume \(x^*\) is not weakly Pareto optimal. This implies that there exists an alternative \(\bar{x} \in \mathcal{X}\) such that \(f_i(\bar{x}) < f_i(x^*)\) for all \(i \in \mathcal{O}\). We thus have that \(w_j^T f(\bar{x}) < w_j^T f(x^*)\) for any \(w_j\) with \(w_j^{(i)} \geq 0\) and \(\sum_{i \in \mathcal{O}} w_j^{(i)} = 1\). Consequently, \(d_j(\bar{x}) < d_j(x^*)\) for all \(j \in \mathcal{S}\). From previous Lemma we also have that \(\langle d(\bar{x}) \rangle_{\alpha}^m < \langle d(x^*) \rangle_{\alpha}^m\). We thus have that the alternative \(\bar{x}\) cannot exist and we have a contradiction.
Some Info:
U.S. Farm Animals Produce 2 Times the Amount of Waste of Entire Human Population
Single Dairy Cow Generates 20 tons of Waste/year
There are 9 Million Cows in the U.S.
From EPA: 2,000 Farms Could Support Biogas from Waste (Less than 200 Installations)

Challenges:
How to Reconcile Priorities (Emissions/Water/Health/Investment/Not-in-my-Backyard)?
How to Derive Fair Incentives/Regulations?
Localización de Instalaciones de Biogas - BioGas Facility Location

\[
\max E = \sum_{j \in \mathcal{F}} E_j^P - \sum_{i \in \mathcal{F}, j \in \mathcal{B}} E_{i,j}^T - \sum_{j \in \mathcal{F}} E_j^U
\]

**Emissions**

\[
\max C = c_i - C^I - C^O - C^T
\]

**Economics**

\[
E^T = \sum_{i \in \mathcal{F}} \sum_{j \in \mathcal{B}} \alpha_{CO_2 Diesel} T_{i,j} d_{i,j}
\]

**Transportation**

\[
T_{i,j} = \frac{S_{i,j}}{\bar{S}}, \quad i \in \mathcal{F}, j \in \mathcal{B}
\]

**Round Trips**

\[
E^P = \sum_{j \in \mathcal{F}} \alpha_{CO_2 CH_4} \cdot \alpha_{CH_4 H} H_j^P
\]

**Processed Waste**

\[
E^U = \sum_{j \in \mathcal{F}} \alpha_{CO_2 CH_4} \cdot \alpha_{CH_4 H} H_j^U
\]

**Unprocessed Waste**

\[
C^I = \sum_{i \in \mathcal{T}} \sum_{j \in \mathcal{B}} c_i \cdot y_{i,j}
\]

**Investment**

\[
C^O = \sum_{i \in \mathcal{T}} \sum_{j \in \mathcal{B}} c_i^O \cdot W_{i,j}
\]

**Processing Cost**

\[
C^T = \sum_{i \in \mathcal{F}} \sum_{j \in \mathcal{B}} c_{i,j}^T \cdot S_{i,j}
\]

**Transportation Cost**

\[
c^{e^-} = \sum_{i \in \mathcal{T}} \sum_{j \in \mathcal{B}} c_i^O \cdot c^{e^-} G_{i,j}
\]

**Electricity Profit**

\[
\sum_{j \in \mathcal{B}} S_{i,j} \leq \bar{F}_i, \quad i \in \mathcal{F}
\]

**Balances**

\[
\sum_{i \in \mathcal{F}} S_{i,j} = \sum_{i \in \mathcal{T}} W_{i,j}, \quad j \in \mathcal{B}
\]

\[
W_i^P = \sum_{j \in \mathcal{B}} S_{i,j}, \quad i \in \mathcal{F}
\]

\[
W_i^U = \bar{F}_i - \sum_{j \in \mathcal{B}} S_{i,j}, \quad i \in \mathcal{F}
\]

\[
G_{i,j} = \alpha_{GW} \cdot W_{i,j}, \quad i \in \mathcal{T}, j \in \mathcal{B}
\]

\[
G_{i,j} \leq \bar{G}_{i,y_{i,j}}, \quad i \in \mathcal{T}, j \in \mathcal{B}
\]
Localización de Instalaciones de Biogas - BioGas Facility Location

![Graph showing emissions vs. cost with CVaR thresholds highlighted]

- CVaR (80%)
- CVaR (100%) (Worst-Case)
- CVaR (0%) (Mean)
Some Info:
CHP Uses Heat Recovery to Simultaneously Provide Electricity, Heating, and Cooling
CHP Efficiency 70-80% vs. Traditional Power Plant Efficiency 40-50%
U.S. CHP Capacity To Increase from 80GW to 120 GW in 10 Years

Design Challenges:
Capture Dynamic Patterns of Electricity/Cooling/Heating Demands
Many Emerging Technologies with Strong Trade-Offs (Investment, Emissions, Water)
Case Study in Pacific Coast of Mexico:
Real Energy Demands & Weather Data for Housing Complex
Housing Complex with 420 Units and 2,400 Inhabitants
## CHP Units

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<thead>
<tr>
<th>Stakeholder</th>
<th>Cost</th>
<th>Emissions</th>
<th>Water</th>
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<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
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<tr>
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<tr>
<td>C</td>
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<tr>
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<tr>
<td>J</td>
<td>2/3</td>
<td>1/3</td>
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</table>

### Costs, Emissions, and Water Usage

<table>
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<tr>
<th></th>
<th>Cost (USD/yr)</th>
<th>CO₂ (Ton/yr)</th>
<th>Water (Kg/yr)</th>
<th>CHP Tech</th>
<th>CHP Size (kWe)</th>
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Medidas de Riesgo Coherentes – Coherent Risk Measures

Coherency Conditions:

- **Homogeneity:** $\rho(\lambda X) = \lambda \rho(X)$
- **Subadditivity:** $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$
- **Normalized:** $\rho(0) = 0$
- **Monotonicity:** If $X_1 \leq X_2$ a.s. then $\rho(X_1) \leq \rho(X_2)$

Norm Conditions:

- **Homogeneity:** $\rho(\lambda x) = \lambda \rho(x)$
- **Subadditivity:** $\rho(x_1 + x_2) \leq \rho(x_1) + \rho(x_2)$
- **Normalized:** $\rho(0) = 0$

Incoherent Risk Measures:

- **Value at Risk:** $\text{VaR}_{\alpha}(X) := \inf_{t \in \mathbb{R}} \{ t : \Pr(X \leq t) \geq \alpha \}$
- **Mean-Standard-Deviation:** $M-SD_\lambda = \mathbb{E}[X] + \lambda \sigma(X)^2$

(Violates Subadditivity)

(Violates Monotonicity)

Coherent Risk Measures:

- **Expected Value:** $\mathbb{E}[X]$
- **Worst-Case Value:** $\text{ess sup}(X)$
- **Conditional Value at Risk:** $\inf_{t \in \mathbb{R}} \left\{ t + \frac{1}{1-\alpha} \mathbb{E}[(X - t)_+] \right\}$
- **Entropic Value at Risk:** $\inf_{t > 0} \left\{ \frac{1}{t} \log \mathbb{E}[\exp(tX)] \right\}$

Some Relationships:

$\mathbb{E}[X] \leq \text{CVaR}_\alpha(X) \leq \text{EVaR}_\alpha(X) \leq \text{ess sup}(X)$

$\text{VaR}_\alpha(X) \leq \text{CVaR}_\alpha(X) \leq \text{EVaR}_\alpha(X)$
Índice de Entropía Generalizada - Generalized Entropy Index

Generalized Entropy Index

\[ GE_\beta(x) := \frac{1}{m\beta(\beta - 1)} \sum_{i \in S} \left( \left( \frac{s_i(x)}{\bar{s}(x)} \right)^\beta - 1 \right), \quad \beta \in [-1, 2] \]

\[ = \frac{1}{m\beta(\beta - 1)} \frac{1}{\bar{s}(x)^\beta} \sum_{i \in S} \left( s_i(x)^\beta - \bar{s}(x)^\beta \right) \]

Mean Log Deviation \( \beta = 0 \)

\[ GE_0(x) = \log \bar{s}(x) - \frac{1}{m} \sum_{i \in S} \log s_i(x) \]

Theil Index \( \beta = 1 \)

\[ GE_1(x) = \frac{1}{m} \sum_{i \in S} s_i(x) \frac{\log s_i(x)}{\bar{s}(x)} \]

\[ = \frac{1}{\bar{s}(x)} \left( \frac{1}{m} \sum_{i \in S} s_i(x) \log s_i(x) - \bar{s}(x) \log \bar{s}(x) \right) \]

Squared Coefficient of Variation \( \beta = 2 \)

\[ GE_2(x) = \frac{1}{2} \left( \frac{\sigma(x)}{\bar{s}(x)} \right)^2 \]
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