Alternating Direction Augmented Lagrangian Algorithms for Convex Optimization

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Alternating direction augmented Lagrangian methods

- Long history: goes back to Gabay and Mercier, Glowinski and Marrocco, Lions and Mercier, and Passty etc.
- Optimization models from partial differential equation
- Maximal monotone operators
- Variational inequalities
- Nonlinear convex optimization
- Linear programming
- Nonsmooth ℓ_1-minimization, compressive sensing

Outline

Outline:

1. ADAL methods for minimizing a convex function that can be written as a sum of simpler functions
2. ADAL methods for solving SDPs

Introduction

SUM-K	SUM-2
$\min F(x) \equiv \sum_{i=1}^{K} f_i(x)$	$\min F(x) \equiv f(x) + g(x)$

- Minimize the sum of convex functions
- Assume the following problem is easy

$$\min_x \quad \tau f_i(x) + \frac{1}{2}\|x - y\|^2$$

- Examples of f_i: $\|x\|_1$, $\|x\|_2$, $\|Ax - b\|^2$, $\|X\|_*$, $-\log\det(X)$, ...

Examples

- Compressed sensing:

$$\min \quad \rho\|x\|_1 + \frac{1}{2}\|Ax - b\|_2^2$$

- Matrix Rank Min:

$$\min \quad \rho\|X\|_* + \frac{1}{2}\|\mathcal{A}(X) - b\|_2^2$$

- Robust PCA:

$$\min_{X,Y} \quad \|X\|_* + \rho\|Y\|_1 : X + Y = M$$

- Sparse Inverse Covariance Selection:

$$\min -\log\det(X) + \langle \Sigma, X\rangle + \rho\|X\|_1$$

Variable Splitting

$$(SUM-2) \quad \min \quad f(x)+g(x)$$

- Variable splitting

$$\begin{array}{ll} \min & f(x)+g(y) \\ \text{s.t.} & x=y \end{array}$$

- Augmented Lagrangian function:

$$\mathcal{L}(x,y;\lambda) := f(x)+g(y)-\langle \lambda, x-y\rangle + \frac{1}{2\mu}\|x-y\|^2$$

- Augmented Lagrangian Method:

$$\begin{cases} (x^{k+1},y^{k+1}) & := \quad \arg\min_{(x,y)} \mathcal{L}(x,y;\lambda^k) \\ \lambda^{k+1} & := \quad \lambda^k - (x^{k+1}-y^{k+1})/\mu \end{cases}$$

Alternating Direction Augmented Lagrangian (ADAL)

- $\mathcal{L}(x, y; \lambda) := f(x) + g(y) - \langle \lambda, x - y \rangle + \frac{1}{2\mu}\|x - y\|^2$
- Solve augmented Lagrangian subproblem alternatingly

$$\begin{cases} x^{k+1} & := & \arg\min_x \mathcal{L}(x, y^k; \lambda^k) \\ y^{k+1} & := & \arg\min_y \mathcal{L}(x^{k+1}, y; \lambda^k) \\ \lambda^{k+1} & := & \lambda^k - (x^{k+1} - y^{k+1})/\mu \end{cases}$$

Symmetric ADAL

- $\mathcal{L}(x, y; \lambda) := f(x) + g(y) - \langle \lambda, x - y \rangle + \frac{1}{2\mu}\|x - y\|^2$
- Symmetric version

$$\begin{cases} x^{k+1} & := & \arg\min_x \mathcal{L}(x, y^k; \lambda^k) \\ \lambda^{k+\frac{1}{2}} & := & \lambda^k - (x^{k+1} - y^k)/\mu \\ y^{k+1} & := & \arg\min_y \mathcal{L}(x^{k+1}, y; \lambda^{k+\frac{1}{2}}) \\ \lambda^{k+1} & := & \lambda^{k+\frac{1}{2}} - (x^{k+1} - y^{k+1})/\mu \end{cases}$$

- Optimality conditions lead to (assuming f and g are smooth)

$$\lambda^{k+\frac{1}{2}} = \nabla f(x^{k+1}), \qquad \lambda^{k+1} = -\nabla g(y^{k+1})$$

Alternating Linearization Method

$$(SUM-2) \quad \min F(x) \equiv f(x) + g(x)$$

- Define

$$Q_g(u,v) := f(u) + g(v) + \langle \nabla g(v), u - v \rangle + \frac{1}{2\mu}\|u - v\|^2$$

$$Q_f(u,v) := f(u) + \langle \nabla f(u), v - u \rangle + \frac{1}{2\mu}\|u - v\|^2 + g(v)$$

- Alternating Linearization Method (ALM)

$$\begin{cases} x^{k+1} & := & \arg\min_x Q_g(x, y^k) \\ y^{k+1} & := & \arg\min_y Q_f(x^{k+1}, y) \end{cases}$$

- Proposed by Kiwiel, Rosa and Ruszczynski for stochastic programming.
- Gauss-Seidel like algorithm

Complexity Bound of ALM

Theorem (Goldfarb, Ma and Scheinberg, 2009)

Assume ∇f and ∇g are Lipschitz continuous with constants $L(f)$ and $L(g)$. For $\mu \leq 1/\max\{L(f), L(g)\}$, ALM satisfies

$$F(y^k) - F(x^*) \leq \frac{\|x^0 - x^*\|^2}{4\mu k}$$

Therefore,

- convergence in objective value
- $O(1/\epsilon)$ iterations for an ϵ-optimal solution

- The first complexity result for splitting and alternating direction type methods
- Question: Can we improve the complexity ?

Optimal Gradient Methods

$\min f(x)$ (assuming ∇f is Lipschitz continuous)

- ϵ-optimal solution $f(x) - f(x^*) \leq \epsilon$
- Classical gradient method

$$x^k = x^{k-1} - \tau_k \nabla f(x^{k-1})$$

Complexity $O(1/\epsilon)$

- Nesterov's acceleration technique (1983)

$$\begin{cases} x^k & := & y^{k-1} - \tau_k \nabla f(y^{k-1}) \\ y^k & := & x^k + \frac{k-1}{k+2}(x^k - x^{k-1}) \end{cases}$$

Complexity $O(1/\sqrt{\epsilon})$

- Optimal first-order method; best one can get

ISTA and FISTA (Beck and Teboulle, 2009)

- Assume g is smooth

$$\min \quad F(x) \equiv f(x) + g(x)$$

- Fixed Point Algorithm (Also called ISTA in compressed sensing)

$$x^{k+1} := \arg\min_x Q_f(x, x^k)$$

or equivalently

$$x^{k+1} := \arg\min_x \tau f(x) + \frac{1}{2}\|x - (x^k - \tau \nabla g(x^k))\|^2$$

- Never minimize g
- Iteration complexity: $O(1/\epsilon)$ for an ϵ-optimal solution $(F(x^k) - F(x^*) \leq \epsilon)$

ISTA and FISTA (Beck and Teboulle, 2009)

$$\min F(x) \equiv f(x) + g(x)$$

- Fast ISTA (FISTA)

$$\begin{cases} x^k & := & \arg\min_x \tau f(x) + \frac{1}{2}\|x - (y^k - \tau \nabla g(y^k))\|^2 \\ t_{k+1} & := & \left(1 + \sqrt{1 + 4t_k^2}\right)/2 \\ y^{k+1} & := & x^k + \frac{t_k - 1}{t_{k+1}}(x^k - x^{k-1}) \end{cases}$$

Complexity $O(1/\sqrt{\epsilon})$

Fast Alternating Linearization Method

- ALM

$$\begin{cases} x^{k+1} & := & \arg\min_x Q_g(x, y^k) \\ y^{k+1} & := & \arg\min_y Q_f(x^{k+1}, y) \end{cases}$$

- Accelerate ALM in the same way as FISTA
- Fast alternating linearization method (FALM)

$$\begin{cases} x^k & := & \arg\min_x Q_g(x, z^k) \\ y^k & := & \arg\min_y Q_f(x^k, y) \\ w^k & := & (x^k + y^k)/2 \\ t_{k+1} & := & \left(1 + \sqrt{1 + 4t_k^2}\right)/2 \\ z^{k+1} & := & w^k + \frac{1}{t_{k+1}}(t_k(y^k - w^{k-1}) - (w^k - w^{k-1})) \end{cases}$$

- computational effort at each iteration is almost unchanged

Complexity Bound for FALM

$$\min F(x) \equiv f(x) + g(x)$$

Theorem (Goldfarb, Ma and Scheinberg, 2009)

Assume ∇f and ∇g are Lipschitz continuous with constants $L(f)$ and $L(g)$. For $\mu \leq 1/\max\{L(f), L(g)\}$, FALM satisfies

$$F(y^k) - F(x^*) \leq \frac{\|x^0 - x^*\|^2}{\mu(k+1)^2}$$

Therefore,

- convergence in objective value
- $O(1/\sqrt{\epsilon})$ iterations for an ϵ-optimal solution
- Optimal first-order method

ALM with skipping steps

At k-th iteration of ALM-S:

- $x^{k+1} := \arg\min_x \mathcal{L}_\mu(x, y^k; \lambda^k)$
- If $F(x^{k+1}) > \mathcal{L}_\mu(x^{k+1}, y^k; \lambda^k)$, then $x^{k+1} := y^k$
- $y^{k+1} := \arg\min_y Q_f(y, x^{k+1})$
- $\lambda^{k+1} := \nabla f(x^{k+1}) - (x^{k+1} - y^{k+1})/\mu$

Note that only one function is required to be smooth.

Complexity result of ALM-S

Theorem (Goldfarb and Scheinberg, 2010)

Assume ∇f is Lipschitz continuous. For $\mu \leq 1/L(f)$, the iterates y^k in ALM-S satisfy:

$$F(y^k) - F(x^*) \leq \frac{\|x^0 - x^*\|^2}{2\mu(k + k_{ns})}, \forall k,$$

where k_{ns} is the number of iterations until the k-th for which $F(x^{k+1}) \leq \mathcal{L}_\mu(x^{k+1}, y^k; \lambda^k)$. Thus $O(1/\epsilon)$ iterations to obtain an ϵ-optimal solution.

Similar algorithm can be designed for FALM with $O(1/\sqrt{\epsilon})$ complexity and only one function is required to be smooth.

Complexity result of ADAL

Theorem (Goldfarb and Huang, 2010)

Assume both ∇f and ∇g are Lipschitz continuous. For $\mu \leq 1/\max\{L(f), L(g)\}$, ADAL needs $O(1/\epsilon)$ iterations to obtain an ϵ-optimal solution.

- Thus far we have been unable to develop a fast $O(1/\sqrt{\epsilon})$ version of ADAL.

SUM-K

From P.L.Lions and B.Mercier's 1979 paper on operator splitting

- Generalization from 2 to K is possible, but
- It seems proving convergence for $K \geq 3$ is "difficult".

$$\min \quad F(x) \equiv f(x) + g(x) + h(x)$$

- Define

$$Q_{gh}(u,v,w) := f(u) + g(v) + \langle \nabla g(v), u - v \rangle + \|u - v\|^2/2\mu \\ + h(w) + \langle \nabla h(w), u - w \rangle + \|u - w\|^2/2\mu.$$

$$\begin{cases} x^{k+1} & := & \arg\min Q_{gh}(x, y^k, z^k) \\ y^{k+1} & := & \arg\min Q_{fh}(x^{k+1}, y, z^k) \\ z^{k+1} & := & \arg\min Q_{fg}(x^{k+1}, y^{k+1}, z) \end{cases}$$

- However, no complexity results for Gauss-Seidel like algorithm!

MSA: A Jacobi Type Algorithm

$$\min \quad F(x) \equiv f(x) + g(x) + h(x)$$

- Multiple Splitting Algorithm (MSA)

$$\begin{cases} x^{k+1} & := & \arg\min Q_{gh}(x, w^k, w^k) \\ y^{k+1} & := & \arg\min Q_{fh}(w^k, y, w^k) \\ z^{k+1} & := & \arg\min Q_{fg}(w^k, w^k, z) \\ w^{k+1} & := & (x^{k+1} + y^{k+1} + z^{k+1})/3 \end{cases}$$

- Jacobi type algorithm
- Can be done in parallel
- We have complexity result!

Complexity Bound for MSA

$$\min F(x) \equiv f(x) + g(x) + h(x)$$

Theorem (Goldfarb and Ma, 2009)

Assume ∇f, ∇g and ∇h are Lipschitz continuous with constants $L(f)$, $L(g)$ and $L(h)$. For $\mu \leq 1/\max\{L(f), L(g), L(h)\}$, MSA satisfies

$$\min\{F(x^k), F(y^k), F(z^k)\} - F(x^*) \leq \frac{\|x_0 - x^*\|^2}{\mu k}.$$

Therefore,

- convergence in objective value
- $O(1/\epsilon)$ iterations for an ϵ-optimal solution

Fast Multiple Splitting Algorithm (FaMSA)

$$\begin{cases} x^k & := & \arg\min Q_{gh}(x, w_x^k, w_x^k) \\ y^k & := & \arg\min Q_{fh}(w_y^k, y, w_y^k) \\ z^k & := & \arg\min Q_{fg}(w_z^k, w_z^k, z) \\ w^k & := & (x^k + y^k + z^k)/3 \\ t_{k+1} & := & \left(1 + \sqrt{1 + 4t_k^2}\right)/2 \\ w_x^{k+1} & := & w^k + \frac{1}{t_{k+1}}[t_k(x^k - w^k) - (w^k - w^{k-1})] \\ w_y^{k+1} & := & w^k + \frac{1}{t_{k+1}}[t_k(y^k - w^k) - (w^k - w^{k-1})] \\ w_z^{k+1} & := & w^k + \frac{1}{t_{k+1}}[t_k(z^k - w^k) - (w^k - w^{k-1})] \end{cases}$$

Complexity Bound for FaMSA

$$\min F(x) \equiv f(x) + g(x) + h(x)$$

Theorem (Goldfarb and Ma, 2009)

Assume ∇f, ∇g and ∇h are Lipschitz continuous with constants $L(f)$, $L(g)$ and $L(h)$. For $\mu \leq 1/\max\{L(f), L(g), L(h)\}$, FaMSA satisfies

$$\min\{F(x^k), F(y^k), F(z^k)\} - F(x^*) \leq \frac{4\|x_0 - x^*\|^2}{\mu(k+1)^2}$$

Therefore,

- convergence in objective value
- $O(1/\sqrt{\epsilon})$ iterations for an ϵ-optimal solution
- optimal first-order method

Comparison of ALM/FALM and MSA/FaMSA

ALM/FALM

- Gauss-Seidel like algorithms
- expected to be faster than MSA/FaMSA since the information from current iteration is used
- complexity results for (SUM-2), no results for (SUM-K) when $K \geq 3$
- only one function needs to be smooth

MSA/FaMSA

- Jacobi like algorithms
- can be done in parallel
- complexity results for (SUM-K) for any $K \geq 2$

ALM and FALM for Robust PCA

- Robust PCA: $f(X) = \|X\|_*, \quad g(Y) = \rho\|Y\|_1$

$$\min_{X,Y\in\mathbb{R}^{m\times n}} \quad \{\|X\|_* + \rho\|Y\|_1 : X + Y = M\}$$

- Subproblem wrt X (a matrix shrinkage operator, corresponds to an SVD):

$$X^{k+1} := \arg\min_X f(X) + g(Y^k) + \langle\nabla g(Y^k), M - X - Y^k\rangle + \|X + Y^k - M\|_F^2/2\mu$$

- Subproblem wrt Y (a vector shrinkage operator):

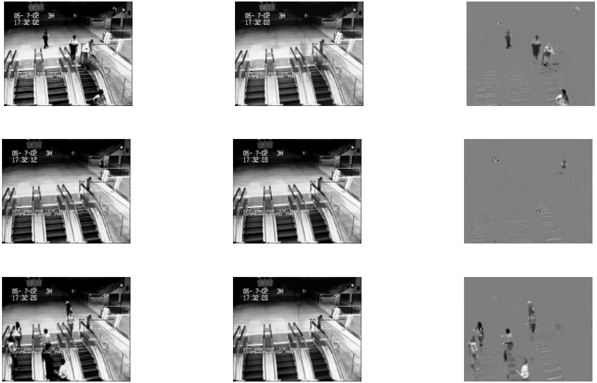
$$Y^{k+1} := \arg\min_Y f(X^{k+1}) + \langle\nabla f(X^{k+1}), M - X^{k+1} - Y\rangle + \|X^{k+1} + Y - M\|_F^2/2\mu + g(Y)$$

Numerical Results

- Surveillance video

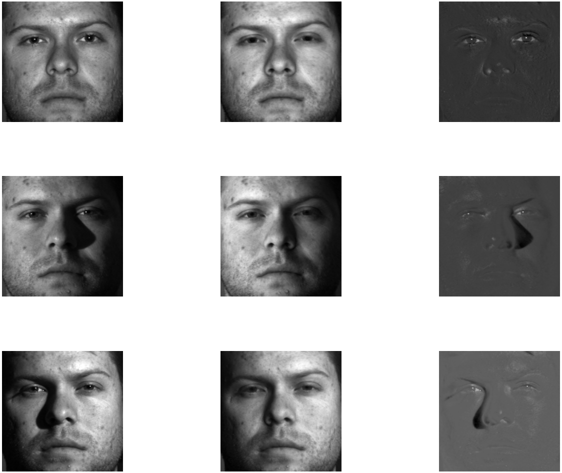
Numerical Results (cont.)

- Surveillance video



Numerical Results (cont.)

- Shadow and specularities removal from face images



Numerical Results (cont.)

- Video denoising

Summary of Numerical Results

Problem	n_1	n_2	SVDs	CPU
Hall	25344	200	43	04:03
Escalator	20800	300	45	05:53
Campus (color)	20480	960	46	01:01:01
Face	40000	65	42	01:39
Video denoising	25344	900	44	01:12:49

Conclusion and future work

Our contribution

- Alternating linearization and multiple splitting methods
- Optimal first-order methods
- First complexity results for splitting and alternating direction methods

Current and Future work

- Extension of ALM/FALM, MSA/FaMSA to constrained problems
- Extension of MSA/FaMSA to nonsmooth problems
- Development of Fast ADAL with $O(1/\sqrt{(}\epsilon))$ complexity
- Line search variants
- Applications in many fields such as Medical Imaging, Machine Learning, Model Selection, Optimal acquisition basis selection (radar), etc.

$$(D) \quad \begin{cases} \min_{y\in\mathbb{R}^m, S\in S^n} & -b^\top y \\ \text{s.t.} & \mathcal{A}^*(y) + S = C, \quad S \succeq 0, \end{cases}$$

- $\mathcal{A}(X) := \left(\left\langle A^{(1)}, X\right\rangle, \cdots, \left\langle A^{(m)}, X\right\rangle\right)^\top$
- $\mathcal{A}^*(y) := \sum_{i=1}^m y_i A^{(i)}$
- Let $A := \left(\mathrm{vec}(A^{(1)}), \cdots, \mathrm{vec}(A^{(m)})\right)^\top \in \mathbb{R}^{m\times n^2}$.
- Constraints: $\mathcal{A}(X) = b \Longrightarrow A\mathrm{vec}(X) = b$
- Operators related to $\mathcal{A}(\cdot)$:

$$\mathcal{A}(\mathcal{A}^*(y)) := (AA^\top)y, \quad \mathcal{A}^*(\mathcal{A}(X)) := \mathrm{mat}\left(\left(A^\top A\right)\mathrm{vec}(X)\right)$$

- Assumption: The matrix A has full row rank and the Slater condition holds for the primal SDP.

Augmented Lagrangian method

Augmented Lagrangian function:

$$\mathcal{L}_\mu(X,y,S) = -b^\top y + \langle X, \mathcal{A}^*(y) + S - C\rangle + \frac{1}{2\mu}\|\mathcal{A}^*(y)+S-C\|_F^2.$$

Algorithmic framework:

- compute y^{k+1} and S^{k+1} at k-th iteration

$$(DL) \qquad \min_{y\in\mathbb{R}^m, S\in S^n} \mathcal{L}_\mu(X^k,y,S), \qquad \text{s.t.} \qquad S \succeq 0.$$

- update the primal variable X^{k+1} by

$$X^{k+1} := X^k + \frac{\mathcal{A}^*(y^{k+1}) + S^{k+1} - C}{\mu}.$$

Pros and Cons:

- Pros: rich theory, well understood and a lot of algorithms
- Cons: $\mathcal{L}_\mu(X^k,y,S)$ is not separable in y and S, and the subproblem (DL) is difficult to minimize

Alternating direction method

- Divide variables into different blocks according to their roles
- Minimize the augmented Lagrangian function with respect to one block at a time while all other blocks are fixed

Framework

$$
\begin{aligned}
y^{k+1} &:= \arg\min_{y \in \mathbb{R}^m} \mathcal{L}_\mu(X^k, y, S^k), \\
S^{k+1} &:= \arg\min_{S \in S^n} \mathcal{L}_\mu(X^k, y^{k+1}, S), \quad S \succeq 0, \\
X^{k+1} &:= X^k + \frac{\mathcal{A}^*(y^{k+1}) + S^{k+1} - C}{\mu}.
\end{aligned}
$$

Alternating direction method

Solving subproblem: $y^{k+1} = \arg\min_{y\in\mathbb{R}^m} \mathcal{L}_\mu(X^k, y, S^k)$

- First-order optimality condition:

$$\nabla_y \mathcal{L}_\mu := \mathcal{A}(X^k) - b + \frac{1}{\mu}\mathcal{A}(\mathcal{A}^*(y^{k+1}) + S^k - C) = 0.$$

- Since $\mathcal{A}\mathcal{A}^*$ is invertible,

$$y^{k+1} := y(S^k, X^k) := -(\mathcal{A}\mathcal{A}^*)^{-1}\left(\mu(\mathcal{A}(X^k) - b) + \mathcal{A}(S^k - C)\right).$$

- Compute $\mathcal{A}\mathcal{A}^*$ prior to executing the algorithm.
- The operator $\mathcal{A}\mathcal{A}^*$ is the identity for many SDP relaxations of combinatorial optimization problems, such as the maxcut, maximum stable set and bisection problems

Alternating direction method

Solving subproblem $S^{k+1} = \arg\min_{S \in S^n} \mathcal{L}_\mu(X^k, \ y^{k+1}, \ S), S \succeq 0$

- Equivalent formulation:

$$\min_{S \in S^n} \quad \left\| S - V^{k+1} \right\|_F^2, \quad S \succeq 0,$$

  where $V^{k+1} := V(S^k, X^k) = C - \mathcal{A}^*(y(S^k, X^k)) - \mu X^k$.

- Hence, the solution is

$$S^{k+1} := V_\dagger^{k+1} := Q_\dagger \Sigma_+ Q_\dagger^\top$$

  where

$$V^{k+1} = Q \Sigma Q^\top = \begin{pmatrix} Q_\dagger & Q_\ddagger \end{pmatrix} \begin{pmatrix} \Sigma_+ & \mathbf{0} \\ \mathbf{0} & \Sigma_- \end{pmatrix} \begin{pmatrix} Q_\dagger^\top \\ Q_\ddagger^\top \end{pmatrix}$$

Alternating direction method

Updating the Lagrange multiplier X^{k+1}

- Updating formula:

$$X^{k+1} := X^k + \frac{\mathcal{A}^*(y^{k+1}) + S^{k+1} - C}{\mu}$$

- Equivalent formulation:

$$X^{k+1} = \frac{1}{\mu}(S^{k+1} - V^{k+1}) = \frac{1}{\mu} V_{\ddagger}^{k+1},$$

  where $V_{\ddagger}^{k+1} := -Q_{\ddagger}\Sigma_{-}Q_{\ddagger}$.

- Note that X^{k+1} is also the optimal solution of

$$\min_{X \in S^n} \quad \left\|\mu X + V^{k+1}\right\|_F^2, \quad X \succeq 0.$$

Convergence results

- Let $\mathcal{P}(V) := \left(V_{\dagger}, V_{\ddagger}\right)$
- Fixed point formulation:

$$\begin{aligned} y^{k+1} &= y(S^k, X^k) \\ \left(S^{k+1}, \mu X^{k+1}\right) &= \mathcal{P}(V^{k+1}) = \mathcal{P}(V(S^k, X^k)). \end{aligned}$$

- Equivalence of the optimality conditions:
  - (X, y, S) satisfies the KKT optimality conditions

$$\mathcal{A}(X) = b, \quad \mathcal{A}^*(y) + S = C, \quad SX = 0, \quad X \succeq 0, \quad Z \succeq 0.$$

  - (X, y, S) satisfies

$$y = y(S, X) \text{ and } \left(S, \mu X\right) = \mathcal{P}(V(S, X)).$$

- Main result: the sequence $\{(X^k, y^k, S^k)\}$ generated by our algorithm from any starting point converges to a solution $(X^*, y^*, S^*) \in \mathcal{X}^*$, where $\mathcal{X}^*$ is the set of primal and dual solution pairs. No complexity results currently known.

Convergence results

- For any $V, \widehat{V} \in S^n$, $\left\| \mathcal{P}(V) - \mathcal{P}(\widehat{V}) \right\|_F \leq \|V - \widehat{V}\|_F$, with equality holding if and only if $V_\dagger^\top \widehat{V}_\ddagger = 0$ and $V_\ddagger^\top \widehat{V}_\dagger = 0$.
- For any $S, X, \widehat{S}, \widehat{X} \in S_+^n$,

$$\|V(S, X) - V(\widehat{S}, \widehat{X})\|_F \leq \left\| \left(S - \widehat{S}, \mu(X - \widehat{X}) \right) \right\|_F$$

  with equality holding if and only if

$$V - \widehat{V} = (S - \widehat{S}) - \mu(X - \widehat{X}).$$

- Let (X^*, y^*, S^*), be an optimal solution. If

$$\|\mathcal{P}(V(S, X)) - \mathcal{P}(V(S^*, X^*))\|_F = \left\| (S - S^*, \mu(X - X^*)) \right\|_F,$$

  then, $(S, \mu X)$ is a fixed point, and hence, (X, y, S) is a primal and dual optimal solution pair.

An expanded problem

- Primal problem:

$$\begin{aligned} \min \quad & \langle C, X \rangle , \\ \text{s.t.} \quad & \mathcal{A}(X) = b, \quad \mathcal{B}(X) \geq d, \\ & X \succeq 0, \quad X \geq 0, \end{aligned}$$

where $\mathcal{B}(X) := \left(\left\langle B^{(1)}, X \right\rangle, \cdots, \left\langle B^{(q)}, X \right\rangle \right)^\top, B^{(i)} \in S^n$.

- Dual problem

$$\begin{aligned} \min_{y \in \mathbb{R}^m, v \in \mathbb{R}^q, S \in S^n, Z \in S^n} \quad & -b^\top y - d^\top v, \\ \text{s.t.} \quad & \mathcal{A}^*(y) + \mathcal{B}^*(v) + S + Z = C, \\ & v \geq 0, \quad S \succeq 0, \quad Z \geq 0 \end{aligned}$$

A multiple splitting scheme

- Augmented Lagrangian function

$$\mathcal{L}_\mu = -b^\top y - d^\top v + \langle X, \mathcal{A}^*(y) + \mathcal{B}^*(v) + S + Z - C \rangle + \frac{1}{2\mu} \| \mathcal{A}^*(y) + \mathcal{B}^*(v) + S + Z - C \|_F^2,$$

Framework

$$y^{k+1} := \arg\min_{y \in \mathbb{R}^m} \mathcal{L}_\mu(X^k, y, v^k, Z^k, S^k),$$

$$v^{k+1} := \arg\min_{v \in \mathbb{R}^q} \mathcal{L}_\mu(X^k, y^{k+1}, v, Z^k, S^k), \quad v \geq 0,$$

$$Z^{k+1} := \arg\min_{Z \in S^n} \mathcal{L}_\mu(X^k, y^{k+1}, v^{k+1}, Z, S^k), \quad Z \geq 0,$$

$$S^{k+1} := \arg\min_{S \in S^n} \mathcal{L}_\mu(X^k, y^{k+1}, v^{k+1}, Z^{k+1}, S), \quad S \succeq 0,$$

$$X^{k+1} := X^k + \frac{\mathcal{A}^*(y^{k+1}) + \mathcal{B}^*(v^{k+1}) + S^{k+1} + Z^{k+1} - C}{\mu}.$$

Eigenvalue decomposition

- Partial eigenvalue decomposition:
  - X and S share the same set of eigenvectors since $XS = 0$.
  - Only need to compute either $V_{\dagger}^{k}$ or $V_{\ddagger}^{k}$.
  - Adaptive scheme (suppose that $V_{\dagger}^{k-1}$ is known)
    - $\kappa_{+}(V^{k-1})$: the total number of the pos. eigenvalues of V^{k-1}.
    - compute $V_{\dagger}^{k}$ for S^{k} if $\kappa_{+}(V^{k-1}) \leq \frac{n}{2}$
    - otherwise, compute $V_{\ddagger}^{k}$ for X^{k}.
- Direct and iterative methods
  - "DSYEVX" in LAPACK and "ARPACK"
- Warm start techniques
  - $V_{\dagger}^{k}$ is the optimal solution of $\min_{S \succeq 0} \|S - V^{k}\|_F^2$
  - Nonlinear programming approaches:

$$R^{k+1} := \arg \min_{R \in \mathbb{R}^{n \times \kappa}} \|RR^\top - V^{k+1}\|_F^2$$

starting from $R := Q_{\dagger}^{k}(\Sigma_{+}^{k})^{\frac{1}{2}}$,

Adjusting penalty parameter μ

- Tune μ so that the primal and dual infeasibilities are balanced
- The primal and dual infeasibilities are proportional to $\frac{1}{\mu}$ and μ, respectively.
- We decrease (increase) μ by a factor γ, $0 < \gamma < 1$, if the primal infeasibility is less than (greater than) a multiple of the dual infeasibility for a number of consecutive iterations.
- μ is required to remain within an interval $[\mu_{\text{min}}, \mu_{\text{max}}]$, where $0 < \mu_{\text{min}} < \mu_{\text{max}} < \infty$.

Other practical issues

- step size for updating X:

$$
\begin{aligned}
X^{k+1} &= X^k + \rho \frac{\mathcal{A}^*(y^{k+1}) + S^{k+1} - C}{\mu} \\
&= (1-\rho)X^k + \frac{\rho}{\mu}(S^{k+1} - V^{k+1}) = (1-\rho)X^k + \rho \bar{X}^{k+1},
\end{aligned}
$$

- termination rule:
  - optimality and feasiblities:

$$
\text{pinf} = \frac{\|\mathcal{A}(X) - b\|_2}{1 + \|b\|_2}, \text{dinf} = \frac{\|C + S + Z - \mathcal{A}^*(y)\|_F}{1 + \|C\|_1}, \text{gap} = \frac{|b^\top y - \langle C, X \rangle|}{1 + |b^\top y| + \langle C, X \rangle}.
$$

  - Stop when $\delta := \max\{\text{pinf, dinf, gap}\} \le \epsilon$
  - Stagnation: "it_stag" is an iteration counter

$$
(\text{it\_stag} > h_1 \text{ and } \delta <= 10\epsilon) \text{ or } (\text{it\_stag} > h_2 \text{ and } \delta <= 10^2\epsilon) \text{ or } (\text{it\_stag} > h_3 \text{ and } \delta <= 10^3\epsilon),
$$

  where $1 < h_1 < h_2 < h_3$ are integers representing different levels of difficulty.

Numerical Results

- Frequency assignment problem
- Maximal stable set problem
- Comparison with SDPNAL (Zhao, Sun and Toh)
  - A semi-smooth Newton-CG method for minimizing the dual augmented Lagrangian function
- Codes were written in C Language MEX-files in MATLAB (Release 7.3.0) and all experiments were performed on a Dell Precision 670 workstation with an Intel Xeon 3.4GHZ CPU and 6GB of RAM.

Numerical Results: frequency assignment problems

- Graph: $G = (V, E)$; Weight matrix $W \in S^r$
- Formulation:

$$\begin{array}{ll} \min & \langle \frac{1}{2k}\mathrm{diag}(We) + \frac{k-1}{2k} W, X \rangle \\ s.t. & X_{i,j} \geq \frac{-1}{k-1}, \quad \forall (i,j) \in E \backslash T, \\ & X_{i,j} = \frac{-1}{k-1}, \quad \forall (i,j) \in T, \\ & \mathrm{diag}(X) = e, \quad X \succeq 0. \end{array}$$

- Operator $\mathcal{A}$: the edges in T and $\mathrm{diag}(X) = e$
- Operator $\mathcal{B}$: the edges in $E \backslash T$
- Scaling: $X_{ij}/\sqrt{2} = \frac{-1}{\sqrt{2}(k-1)}$ and $X_{ij}/\sqrt{2} \geq \frac{-1}{\sqrt{2}(k-1)}$

$$\begin{aligned} y^{k+1} &= -\left(\mu(\mathcal{A}(X^k) - b) + \mathcal{A}(S^k - C)\right) \\ v^{k+1} &= \max\left(-\left(\mu(\mathcal{B}(X^k) - d) + \mathcal{B}(S^k - C)\right), \mathbf{0}\right). \end{aligned}$$

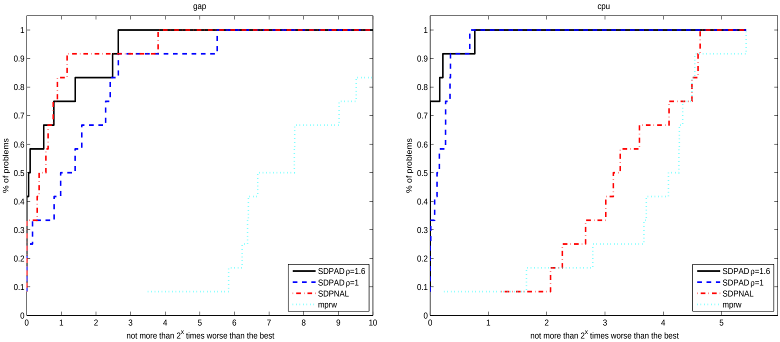
Numerical Results: frequency assignment problems

Table: Computational results

			ADAL					SDPNAL	
name	n	$\hat{m}$	pinf	dinf	itr	gap	cpu	gap	cpu
fap01	52	1378	1.72e-07	3.48e-07	610	7.76e-07	0.35	1.39e-7	6.31
fap02	61	1866	1.24e-06	8.77e-06	1121	9.61e-06	0.83	1.15e-5	4.12
fap03	65	2145	5.89e-06	4.30e-06	840	1.48e-06	0.73	2.27e-6	7.44
fap04	81	3321	1.56e-07	3.10e-07	929	9.58e-06	1.58	1.53e-5	19.69
fap05	84	3570	3.49e-06	3.76e-06	768	8.84e-06	1.29	1.09e-5	31.59
fap06	93	4371	3.17e-06	2.50e-06	568	9.71e-06	1.20	1.73e-5	29.84
fap07	98	4851	4.54e-06	4.95e-06	543	9.89e-06	1.24	5.75e-6	29.88
fap08	120	7260	1.85e-06	1.06e-06	699	4.04e-06	2.60	5.93e-6	25.25
fap09	174	15225	7.99e-07	9.98e-07	505	2.06e-07	4.64	2.86e-6	59.67
fap10	183	14479	3.22e-06	5.46e-06	1250	4.96e-04	14.29	7.93e-5	1:50
fap11	252	24292	3.26e-06	3.21e-06	1654	4.99e-04	48.01	1.89e-4	5:15
fap12	369	26462	4.70e-07	4.94e-07	5000	9.42e-05	6:39	1.60e-4	13:14

Numerical Results: frequency assignment problems

Figure: Performance profiles of two variants of ADAL, SDPNAL and mprw



(a) gap (b) cpu

Numerical Results: computing $\theta(G)$ and $\theta_+(G)$

- Formulation:

$$
\begin{aligned}
\theta(G) := \max \quad & \left\langle ee^\top, X \right\rangle \\
\text{s.t.} \quad & X_{ij} = 0, \quad (i,j) \in E, \quad \langle I, X \rangle = 1, \\
& X \succeq 0,
\end{aligned}
$$

$$
\begin{aligned}
\theta_+(G) := \max \quad & \left\langle ee^\top, X \right\rangle \\
\text{s.t.} \quad & X_{ij} = 0, \quad (i,j) \in E, \quad \langle I, X \rangle = 1, \\
& X \succeq 0, \quad X \geq 0,
\end{aligned}
$$

- Scaling: $X_{ij}/\sqrt{2} = 0$ and $\frac{1}{\sqrt{n}} \langle I, X \rangle = \frac{1}{\sqrt{n}}$.

Numerical Results: computing $\theta(G)$

Table: Computational results on computing $\theta(G)$

			ADAL					SDPNAL	
name	n	$\hat{m}$	pinf	dinf	itr	gap	cpu	gap	cpu
theta102	500	37467	8.76e-7	6.27e-7	256	9.67e-7	47	1.6e-8	50
theta103	500	62516	2.85e-7	9.42e-7	257	3.33e-7	57	4.6e-8	1:00
theta104	500	87245	3.40e-7	9.72e-7	260	6.53e-7	48	7.6e-8	58
theta123	600	90020	3.42e-7	9.71e-7	263	4.57e-7	1:43	4.1e-8	1:34
MANN-a27	378	703	8.45e-7	3.50e-6	503	8.81e-6	27	8.3e-8	07
sanr200-0.7	200	6033	7.83e-7	9.98e-7	219	3.04e-7	04	1.4e-7	04
c-fat200-1	200	18367	1.00e-6	1.76e-7	302	9.15e-7	04	8.5e-8	09
ham-10-2	1024	23041	4.72e-7	3.16e-6	597	9.51e-6	22:8	9.0e-8	02
ham-8-3-4	256	16129	9.92e-8	9.94e-7	199	8.04e-7	05	1.3e-8	10
ham-9-5-6	512	53761	5.70e-7	4.51e-6	1000	1.52e-5	2:48	1.4e-6	1:33
brock400-1	400	20078	9.75e-7	8.06e-7	254	8.30e-7	25	1.7e-8	26
keller4	171	5101	5.06e-7	9.88e-7	249	9.32e-7	03	1.3e-8	05
p-hat300-1	300	33918	7.58e-7	6.81e-7	764	9.96e-7	37	5.3e-7	1:45
G43	1000	9991	2.84e-6	3.91e-6	935	7.68e-6	21:17	4.2e-8	1:33
2dc.512	512	54896	2.24e-5	2.57e-5	1000	2.87e-4	5:51	1.7e-4	32:16
1dc.1024	1024	24064	7.82e-5	5.10e-5	747	2.79e-4	24:26	2.9e-6	41:26
1et.1024	1024	9601	1.93e-4	1.32e-4	603	7.11e-4	20:03	1.8e-6	1:01:14
1tc.1024	1024	7937	4.16e-4	5.21e-4	611	9.93e-4	21:47	2.2e-6	1:48:04
1zc.1024	1024	16641	9.72e-7	3.78e-7	608	4.16e-7	23:15	3.3e-8	4:15
2dc.1024	1024	169163	1.85e-5	2.71e-5	1000	3.39e-4	49:05	9.9e-5	2:57:56
1dc.2048	2048	58368	2.82e-4	2.16e-4	473	9.99e-4	2:50:44	1.5e-6	6:11:11
1et.2048	2048	22529	1.78e-4	4.05e-4	904	9.93e-4	4:54:57	8.8e-7	7:13:55
1tc.2048	2048	18945	1.79e-4	4.48e-4	1000	1.08e-3	5:15:14	7.9e-6	9:52:09
1zc.2048	2048	39425	1.42e-6	3.10e-6	941	9.17e-6	6:39:07	1.2e-6	45:16
2dc.2048	2048	504452	1.88e-5	8.37e-6	1000	3.17e-4	7:12:50	4.4e-5	15:13:19

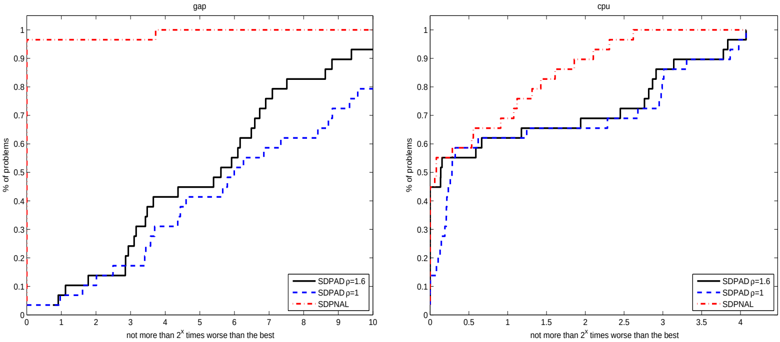
Numerical Results: computing $\theta_+(G)$

Table: Computational results on computing $\theta_+(G)$

			ADAL					SDPNAL	
name	n	$\hat{m}$	pinf	dinf	itr	gap	cpu	gap	cpu
theta102	500	37467	2.94e-7	9.58e-7	281	2.84e-7	1:01	8.4e-8	3:31
theta103	500	62516	3.27e-7	9.52e-7	262	4.85e-7	1:01	2.3e-8	3:28
theta104	500	87245	3.57e-7	9.55e-7	266	7.35e-7	52	1.6e-7	2:35
theta123	600	90020	3.77e-7	9.43e-7	267	6.02e-7	1:39	1.2e-7	6:44
MANN-a27	378	703	8.98e-7	4.08e-6	530	8.17e-6	32	1.6e-7	35
sanr200-0.7	200	6033	8.61e-7	9.57e-7	228	4.47e-7	05	2.9e-7	11
c-fat200-1	200	18367	4.71e-7	2.04e-7	306	8.40e-7	04	2.1e-7	36
ham-8-3-4	256	16129	9.91e-8	9.94e-7	199	8.05e-7	05	2.7e-10	05
ham-9-5-6	512	53761	6.49e-8	7.58e-7	472	3.25e-7	2:19	2.6e-7	42
ham-10-2	1024	23041	4.99e-8	5.49e-7	653	9.78e-7	27:58	4.2e-7	4:35
brock400-1	400	20078	9.98e-7	7.54e-7	258	9.83e-7	29	3.5e-9	1:45
keller4	171	5101	3.61e-6	9.01e-6	331	2.98e-6	05	3.7e-7	43
p-hat300-1	300	33918	1.34e-6	5.93e-7	567	1.50e-6	29	7.9e-7	6:50
G43	1000	9991	4.40e-6	5.45e-6	864	8.02e-6	20:22	2.1e-7	52:00
2dc.512	512	54896	9.43e-6	9.09e-6	1000	8.64e-5	5:10	3.8e-4	2:25:15
1dc.1024	1024	24064	2.13e-5	8.50e-6	1000	5.98e-5	57:20	1.4e-5	5:03:49
1et.1024	1024	9601	1.18e-4	2.16e-4	651	5.26e-4	36:04	1.1e-5	6:45:50
1tc.1024	1024	7937	1.65e-4	2.67e-4	799	3.53e-4	49:13	8.7e-4	10:37:57
1zc.1024	1024	16641	5.71e-5	1.80e-5	506	5.30e-4	28:22	1.6e-7	40:13
2dc.1024	1024	169163	3.81e-6	2.37e-6	1000	3.85e-5	50:34	7.3e-4	11:57:25
1dc.2048	2048	58368	2.01e-4	9.45e-5	517	9.99e-4	4:25:17	9.7e-5	35:52:44
1et.2048	2048	22529	1.76e-4	2.66e-4	659	6.80e-4	5:15:09	4.0e-5	80:48:17
1tc.2048	2048	18945	1.86e-4	3.97e-4	862	6.09e-4	6:35:33	1.4e-3	73:56:01
1zc.2048	2048	39425	3.58e-7	4.60e-6	953	8.27e-6	7:14:21	2.3e-7	2:13:04
2dc.2048	2048	504452	4.84e-6	2.79e-6	1000	4.95e-5	5:45:25	2.7e-3	45:21:42

Numerical Results: computing $\theta(G)$

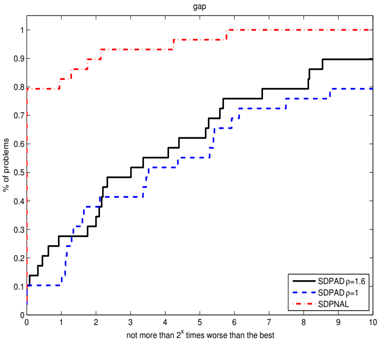
Figure: Performance profiles of two variants of ADAL and SDPNAL



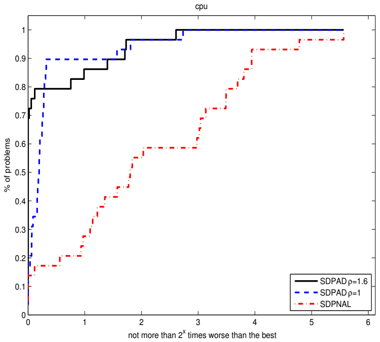
(a) gap (b) cpu

Numerical Results: computing $\theta_+(G)$

Figure: Performance profiles of two variants of ADAL and SDPNAL



(a) gap



(b) cpu

Numerical Results: BIQ problems

- The binary integer quadratic programming problem:

$$\min x^\top Qx, \text{ s.t. } x \in \{0,1\}^n$$

whose SDP relaxation is:

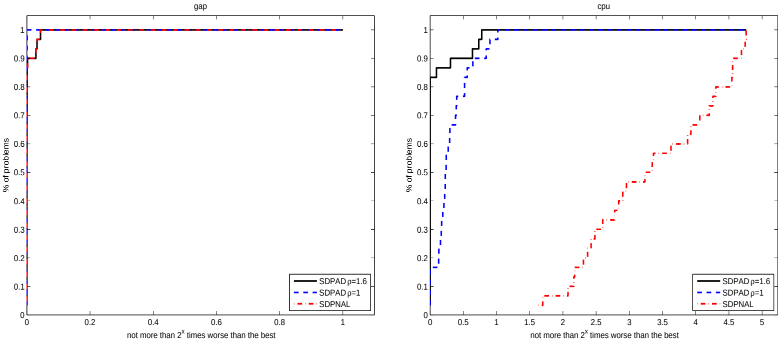
$$\begin{aligned} \min \quad & \left\langle \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, X \right\rangle \\ s.t. \quad & X_{ii} - X_{n,i} = 0, i = 1, \cdots, n-1, \\ & X_{nn} = 1, \quad X \succeq 0, \quad X \geq 0, \end{aligned}$$

- Scaling: $\sqrt{\frac{2}{3}}(X_{ij} - X_{n,i}) = 0$
- "best upper bound" take from "Biq mac library" and

$$\%\text{dgap} \quad := \quad \left| \frac{\text{best upper bound} - \text{dobj}}{\text{best upper bound}} \right| \times 100\%.$$

Numerical Results: BIQ problems

Figure: Performance profiles of two variants of ADAL and SDPNAL



(a) %dgap (b) cpu