# On the Geometry of Lift-and-Project

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# Intersection cuts (1971)

- $x = \bar{x} - \sum_{j \in J} \bar{a}_j s_j, \quad x \in \mathbb{Z}^q_+, s \in \mathbb{R}^n_+,$ | *LP* cone $C \subseteq \mathbb{R}^n_+$

- $S$ : closed convex set such that

  (i) $\bar{x} \in$ int $S$

  (ii) int $S$ contains no feasible integer point

- For $j \in J$, $s^*_j := \max\{s_j : \bar{x} - \bar{a}_j s_j \in S\}$ | Intersect extreme rays of $C$ with bd $S$

**Intersection cut** from $S$ : $\alpha s \geq 1$, where $\alpha_j = \frac{1}{s^*_j}$, $j \in J$, cuts off $\bar{x}$ but no feasible integer point.

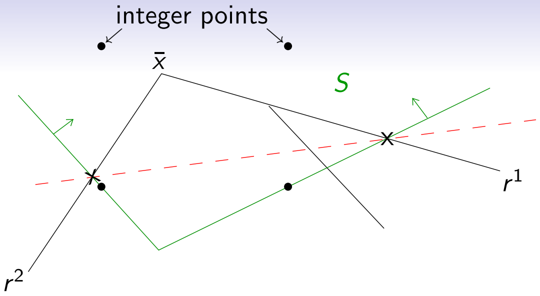



Figure: 1

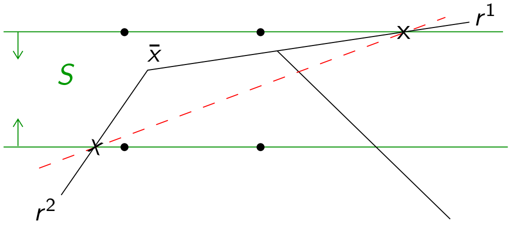



Figure: 2

- Intersection cut from $0 \leq x_k \leq 1$ $\longrightarrow$ Disjunctive cut from $x_k \leq 0 \;\vee\; x_k \geq 1$
- Disjunctive programming: optimization over unions of polyhedra:

$$\begin{pmatrix} \tilde{A}x \geq \tilde{b} \\ -x_k \leq 0 \end{pmatrix} \;\vee\; \begin{pmatrix} \tilde{A}x \geq b \\ x_k \geq 1 \end{pmatrix} \tag{1}$$

- Lift-and-Project:
  convex hull of (1) in $(2m+n)$-space (CGLP)
  is tighter than the LP relaxation

- Optimizing CGLP through pivots in the LP tableau

  replaces

  an intersection cut from $0 \leq x_k \leq 1$, where

$$x_k = a_{k0} - \sum_{j \in J} a_{kj} x_j$$

  with an intersection cut from $0 \leq x_k \leq 1$, where

$$x_k = \tilde{a}_{k0} - \sum_{j \in L} \tilde{a}_{kj} x_j$$

# Lift-and-project executed in the LP tableau

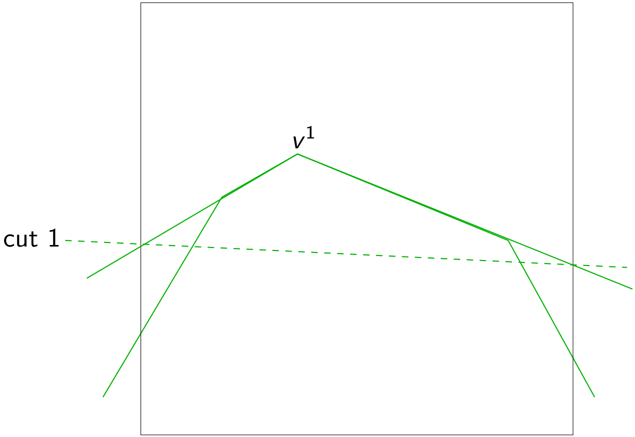
| | | |
|---|---|---|
| • M. Perregaard | XPRESS | 2003 |
| • P. Bonami | COIN-OR | 2006-2009 |
| 3 variants (one with iterative disjunctive modularization) | | |
| • F. Wesselmann | MOPS | 2009 |
| • T. Kis | extension to arbitrary 2-term disjunctions | 2009 |

**Comparison of L&P with GMI cuts**

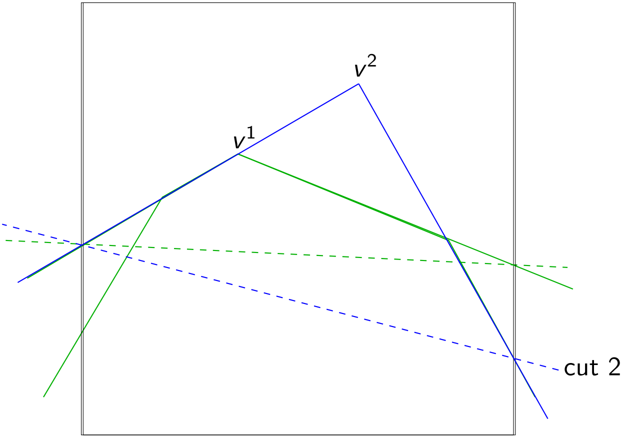
On the hardest 1/3 of instances,

- Gap closed by 10 rounds of cuts — increased by $> 50\%$
- Time for a complete run (B&Bd) — reduced by $> 50\%$

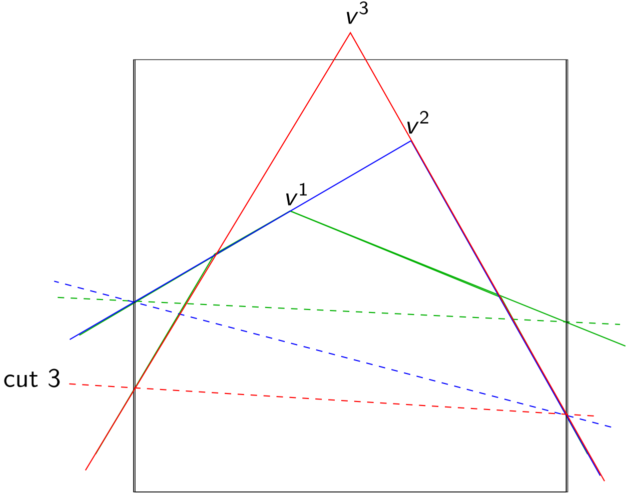
# Intersection cuts corresponding to Lift-and-Project pivot sequence



# Intersection cuts from Infeasible Solutions (MIG Cuts from Infeasible Tableaus)



# Intersection cuts from Infeasible Solutions (MIG Cuts from Infeasible Tableaus)



# Theorem

Let $v$ be the vertex corresponding to the current basic solution

$$x_i = \bar{a}_{i0} - \sum_{j \in J} \bar{a}_{ij} s_j \qquad i \in I$$

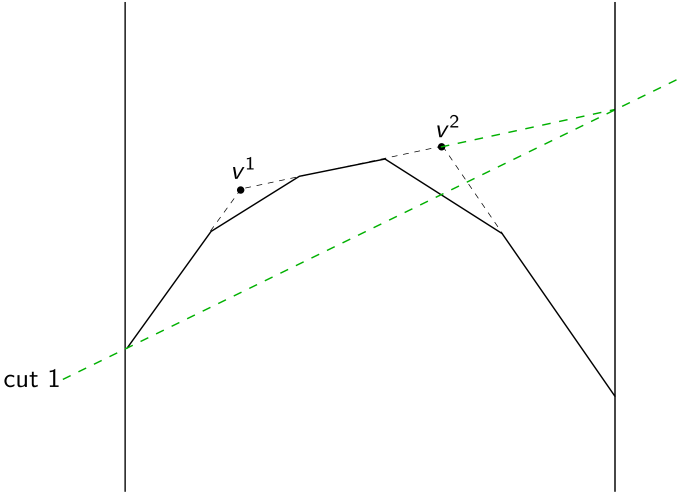
and let $\alpha x \geq 1$ be the intersection cut derived from source row $k$.

Let $v'$ be the vertex corresponding to the solution obtained by a pivot on $\bar{a}_{i_* j_*}$, and let $\alpha' x \geq 1$ be the intersection cut derived from the same source row transformed by the pivot. Then

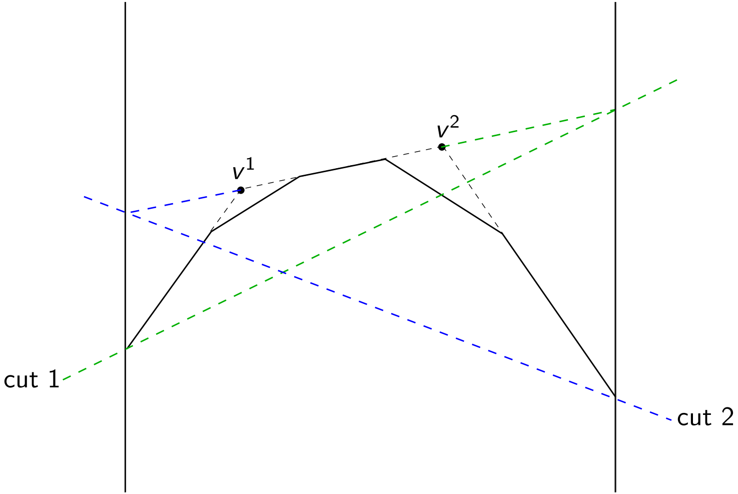
$\alpha' x \geq 1$ strictly dominates $\alpha x \geq 1$ if and only if

(i) the facet of $P$ defined by $s_{j_*} \geq 0$ is a simplex

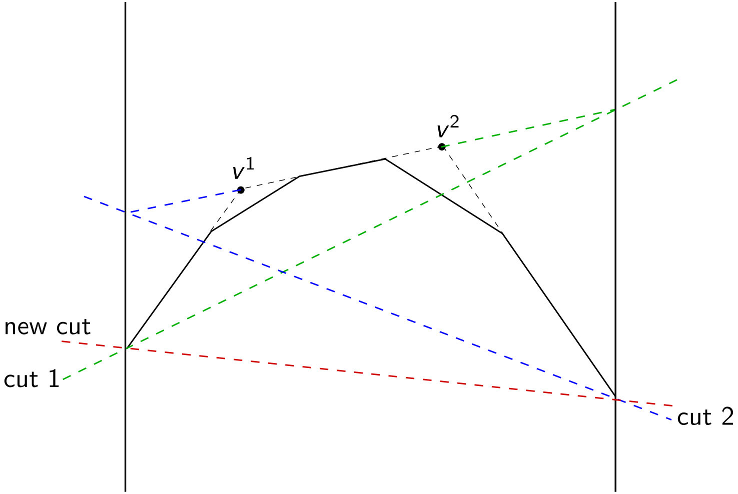
(ii) $s_j$ enters the basis at a negative value ($s_{j_*} = \frac{\bar{a}_{i_* 0}}{\bar{a}_{i_* j_*}} < 0$)



$v^1$
$v^2$
cut 1



$v^1$
$v^2$
cut 1
cut 2


$v^1$
$v^2$
new cut
cut 1
cut 2

# Generalized Intersection Cuts

- Generating intersection-type cuts from non-conic polyhedra

$P := \{x \in \mathbb{R}^n : Ax \geq b,\ x \geq 0\}$

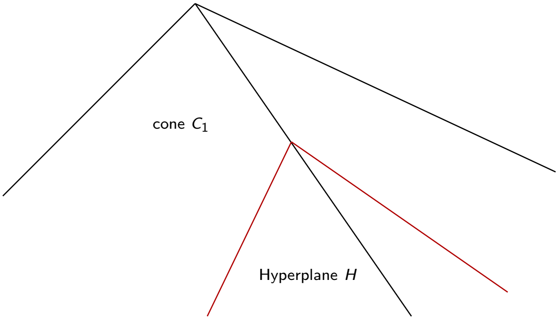
$P_I := \{x \in P : x_j \in \mathbb{Z},\ j \in N_1 \subseteq N\}$

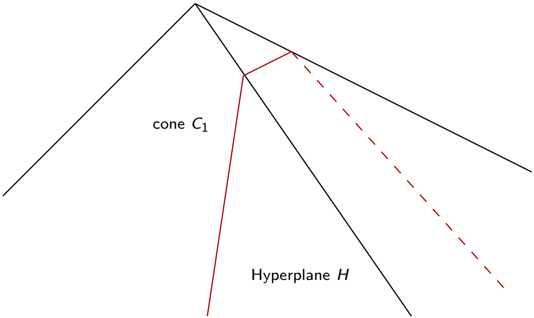
$C_1$ - polyhedral cone with apex at $v^1$, $P \subset C_1$

$S$ - convex set, $v^1 \in \mathrm{int}\, S$, $P_I \cap \mathrm{int}\, S = \emptyset$

$H^+$ - halfspace, facet defining for $P$, $v^1 \in \mathrm{int}\, H^+$

$C := C_1 \cap H^+$ non-conic polyhedron


cone $C_1$
Hyperplane $H$


cone $C_1$
Hyperplane $H$

**Theorem 1.** If $k$ extreme rays of $C_1$, $1 \leq k \leq n-1$, intersect $H$ before $\mathrm{bd}\, S$, then $C$ has $(k+1)(n-k)$ extreme rays.

Let $\alpha^1 x \geq 1$ be the intersection cut from $C_1$ and $S$.

**Assumption.** $S$ is bounded, $q^j \notin H$ for $j = 1, \ldots, q$, and for every edge $e_h$ of $C_1 \cap \{x : \alpha^1 x = 1\}$ intersected by $H$,
$\mathrm{relint}\,(e_h) \cap \mathrm{bd}\, S = \emptyset$

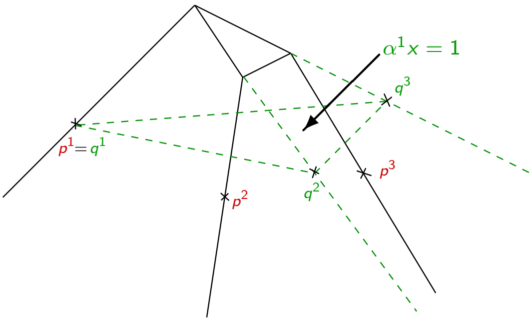
Let $r_j$, $j \in Q$, be the extreme rays of $C$, and

$p^j := r_j \cap \mathrm{bd}\, S$, $j \in Q$.

Let $Q_1 := \{j \in Q : r_j$ is an extreme ray of both $C_1$ and $C\}$,
$Q_2 = Q \setminus Q_1$

**Theorem 2.** Every $p^j$, $j \in Q$, satisfies $\alpha^1 p^j \geq 1$.

Every $p^j$, $j \in Q_2$, satisfies $\alpha^1 p^j > 1$.




$q^1, q^2, q^3$ – intersection points of extreme rays of $C_1$ with $\mathrm{bd}\, S$

$p^1, p^2, p^3$ – intersection points of extreme rays of $C$ with $\mathrm{bd}\, S$

**Theorem 3.** Every basic optimal solution to

$$\min \quad \alpha v^1$$
$$\alpha p^j \geq 1, \quad j \in Q$$

defines a minimal valid inequality $\bar{\alpha}x \geq 1$ for $P_I$ that strictly dominates $\alpha^1 x \geq 1$

**Proof.** (a) (1) has a finite minimum

(b) for every basic optimal $\bar{\alpha}$, $\bar{\alpha}x \geq 1$ is a minimum valid inequality

(c) strict dominance follows from $\bar{\alpha}_j < \alpha^1_j$ for all $j \in Q_2$

# New paradigm for generating cuts

- Instead of storing cuts, we can store the $p^j$ and generate cuts as needed
- Non-iterative (non-recursive) way of generating higher rank cuts
- The polyhedron in $\alpha$ is the reverse polar of $\mathrm{conv}\,\{p^j : j \in Q\}$

# Generating the intersection points $p^j$

$x = \bar{a}_0 - \sum_{j \in J} \bar{a}_j s_j$ basic solution for $C_1 = C(\bar{a}_0)$

$r_j = \{x \in \mathbb{R}^n : \bar{a}_0 - \bar{a}_j s_j,\ s_j \geq 0\},\ j \in J$ extreme rays of $C(\bar{a}_0)$

$H^+ = \{x : x_h \geq 0\}$, $h$ basic with $\bar{a}_{h0} > 0$ ($\bar{a}_0 \in \mathrm{int}\, H^+$)

$r_j \cap H^+ \neq \emptyset \Leftrightarrow \bar{a}_{hj} > 0$

For $j \in J^- := \{j \in J : \bar{a}_{hj} \leq 0\},\ r_j \cap H = \emptyset$ hence $r_j$ is an infinite edge of $C$:

$r_j \cap \mathrm{bd}\, S = p^j = q^j,\ j \in J^-(= Q_1)$

(no new intersection points – no pivots)

For $j \in J^+ := \{j \in J : \bar{a}_{hj} > 0\}$, $r_j \cap H = \bar{a}_0 - \bar{a}_j \tilde{s}_j,\ \tilde{s}_j = \bar{a}_{h0}/\bar{a}_{hj}$

Pivot on $\bar{a}_{hj}$ : $x = \tilde{a}_0 - \sum_{\ell \in J \setminus \{j\}} \tilde{a}_\ell s_\ell - \tilde{a}_h x_h$

New vertex of $C$ : $\tilde{a}_0$, new cone $C(\tilde{a}_0)$

Extreme rays of $C(\tilde{a}_0)$ : $r_\ell := \{x \in \mathbb{R}^n : x = \tilde{a}_0 - \tilde{a}_\ell s_\ell,\ s_\ell \geq 0\}$

$k$ of these $r_\ell$ contain (finite) edges of $C$

$n - k$ are new infinite edges of $C$, with $p^\ell := r_\ell \cap \mathrm{bd}\, S,\ \ell \in Q_2$

Repeating this for every $j \in J^+$ gives $k(n-k)$ new $p^\ell$, $\ell \in Q_2$

(at the cost of $(1+k)(n-k)$ pivots)

# Iterate the procedure to completion

Typical iteration:

- Activate a new hyperplane $H'$ to replace $C$ with $C' := C \cap H'^{+}$
- Generate the new vertices $v'_j$ created by intersecting $H'$ with rays $r_j$ of $C$
- Find the new extreme rays $r_{\ell(j)}$ and their intersection points $p^{\ell(j)}$ with $\mathrm{bd}\, S$
- Update $Q$ by removing those $p^j$ cut off by $H'$ and adding the intersection points $p^{\ell(j)}$ with $\mathrm{bd}\, S$

The procedure is complete when all facets of $P$ have been activated.

At that point $p^{j \in P}$ for all $j$ in the current set.