# A Principled Approach to MILP Modeling

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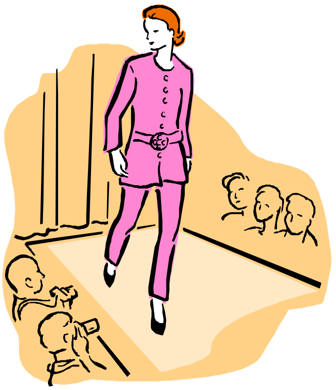
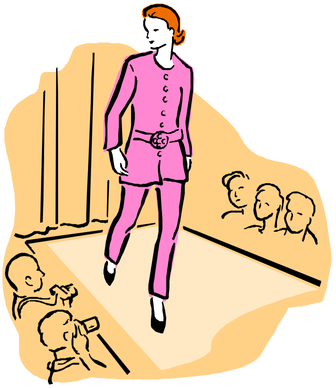
# Proposal



- MILP modeling is an art, but it need not be unprincipled.

# Proposal



- MILP modeling is an art, but it need not be unprincipled.
- It has two basic components:
  - Disjunctive modeling of subsets of continuous space.
  - Knapsack modeling of counting ideas.

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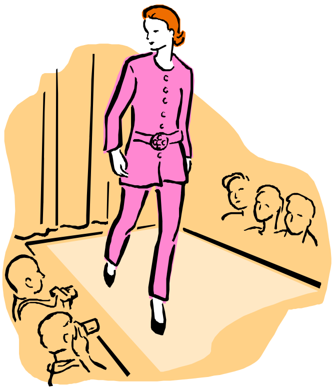

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• It has two basic components:

  • Disjunctive modeling of subsets of continuous space.

  • Knapsack modeling of counting ideas.

• MILPs can model subsets of continuous space that are unions of polyhedra.

  • …that is, represented by disjunctions of linear systems.

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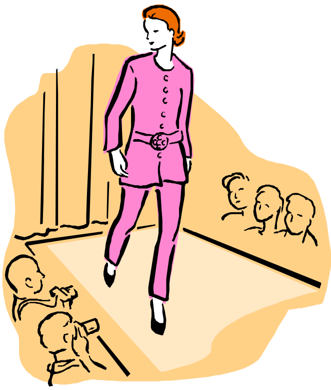


- MILP modeling is an art, but it need not be unprincipled.
- It has two basic components:
  - Disjunctive modeling of subsets of continuous space.
  - Knapsack modeling of counting ideas.
- MILPs can model subsets of continuous space that are unions of polyhedra.
  - …that is, represented by disjunctions of linear systems.
- So a principled approach is to analyze the problem as

| | | |
|---|---|---|
| disjunctions of linear systems | + | integer knapsack inequalities |

# Proposal

- **Jeroslow's Representability Theorem** provides theoretical basis for disjunctive modeling.
  - "Bounded MILP representability" assumes bounded integer variables.
  - This is inadequate for knapsack modeling.

# Proposal

- **Jeroslow's Representability Theorem** provides theoretical basis for disjunctive modeling.
    - "Bounded MILP representability" assumes bounded integer variables.
    - This is inadequate for knapsack modeling.
- We will **generalize** Jeroslow's theorem.
    - Knapsack modeling accommodated.
    - Integer variables can be unbounded.

# Outline

- **Bounded mixed integer representability**
    - Bounded representability theorem.
    - Convex hull formulation
    - *Example:* Fixed charge problem
    - Why the disjunctive model works
    - Multiple disjunctions
    - *Example:* Facility location
    - *Example:* Lot sizing with setup costs
    - Big-M disjunctive formulation
    - *Example:* Health care benefits

# Outline

- **General mixed integer representability**
    - Knapsack models
    - General representability theorem.
    - Convex hull formulation
    - *Example:* Facility location
    - Why a single recession cone
    - *Example:* Freight packing and transfer
    - Research issues

# Bounded MILP Representability

# Bounded representability theorem

Definition of R. Jeroslow:

A subset $S$ of $\mathbb{R}^n$ is
**bounded MILP representable**
if $S$ is the projection onto $x$ of the
feasible set of some MILP
constraint set of the form

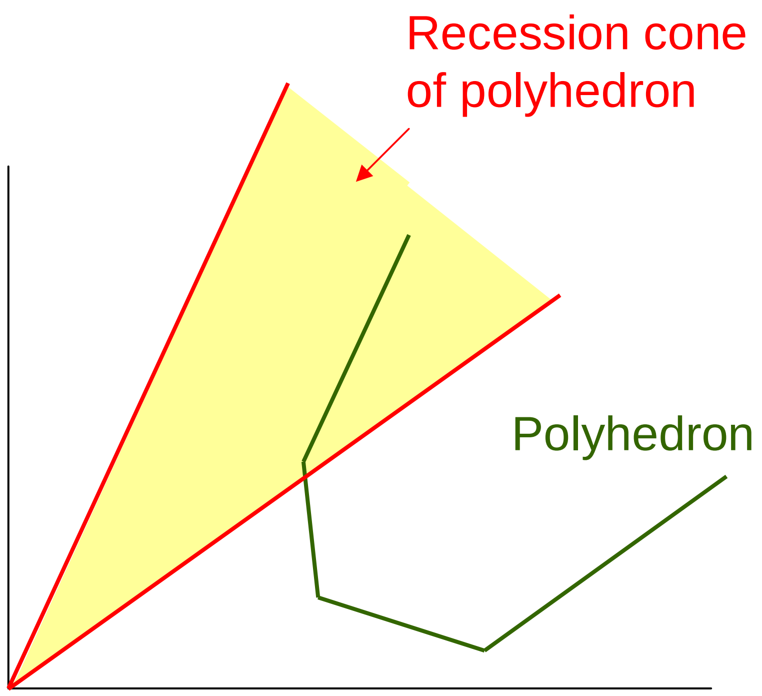
$$Ax + Bz + Dy \geq b$$
$$x \in \mathbb{R}^n, \quad \boxed{z \in \mathbb{R}^m}$$
$$\boxed{y \in \{0,1\}^p}$$

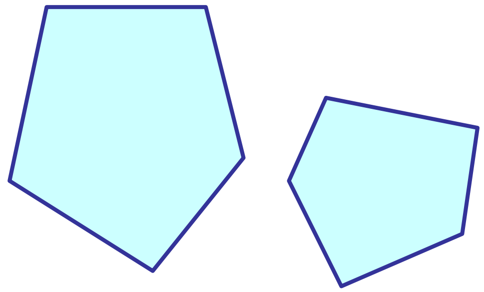
**Bounded** general integer variables can be encoded as 0-1 variables

Auxiliary continuous variables can be used

# Bounded representability theorem

**Theorem** (Jeroslow). A subset of continuous space is bounded MILP representable if and only if it is the union of finitely many polyhedra having the same recession cone.



Union of polyhedra with the same recession cone (in this case, the origin)

# Convex hull formulation

Start with a disjunction of linear systems to represent the union of polyhedra.

$$\bigvee_k \left( A^k x \geq b^k \right)$$

The *k*th polyhedron is $\{x \mid A^k x \geq b\}$

Introduce a 0-1 variable $y_k$ that is 1 when $x$ is in polyhedron $k$.

Disaggregate $x$ to create an $x^k$ for each $k$.

$$A^k x^k \geq b^k y_k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k x^k$$

$$y_k \in \{0,1\}$$

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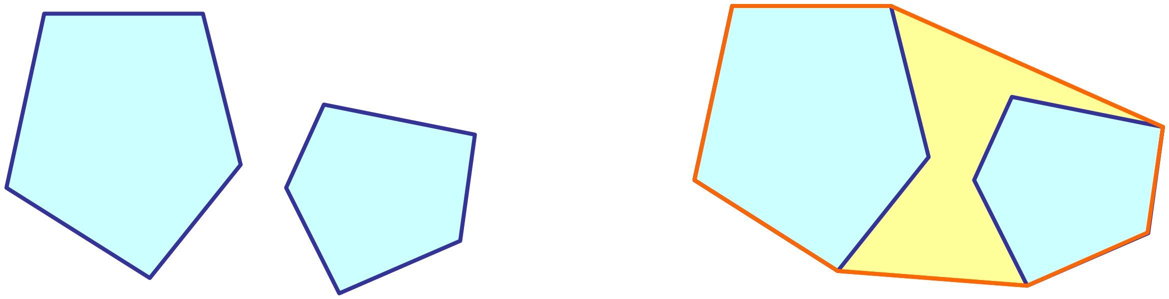
Every bounded MILP representable set has a model of this form.

# Convex Hull Formulation

- The continuous relaxation of this disjunctive MILP provides a **convex hull relaxation** of the disjunction.
    - Strictly, it describes the **closure** of the convex hull.



Union of polyhedra

Convex hull relaxation
(tightest linear relaxation)

# Idea behind the convex hull formulation

Start by formulating a convex hull formulation of the relaxation of the disjunction…

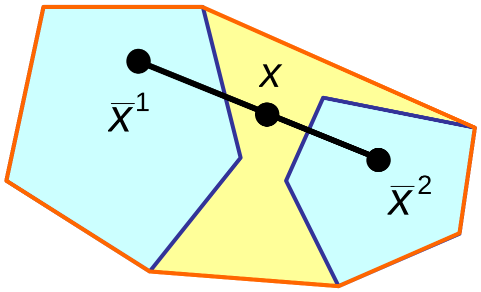
Write each solution as a convex combination of points in the polyhedron

$$A^k \bar{x}^k \geq b^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k y_k \bar{x}^k$$

$$y_k \in [0,1]$$




Convex hull relaxation

# Idea behind the convex hull formulation

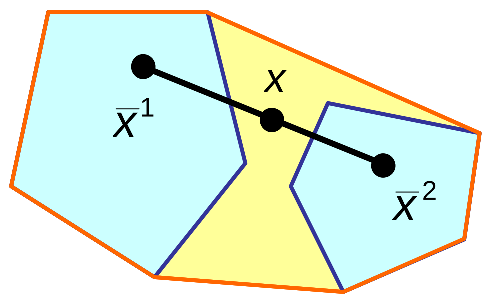
Now apply a change of variable

Write each solution as a convex combination of points in the polyhedron

$$A^k \bar{x}^k \geq b^k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k y_k \bar{x}^k$$
$$y_k \in [0,1]$$

Change of variable
$x = y_k \bar{x}^k$

$$A^k x^k \geq b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in [0,1]$$




Convex hull relaxation

# Idea behind the convex hull formulation

Now make $y_k$s 0-1 variables to get an MILP representation

$$A^k x^k \geq b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in [0,1]$$

Make $y_k$s 0-1

$$A^k \bar{x}^k \geq b^k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k y_k \bar{x}^k$$
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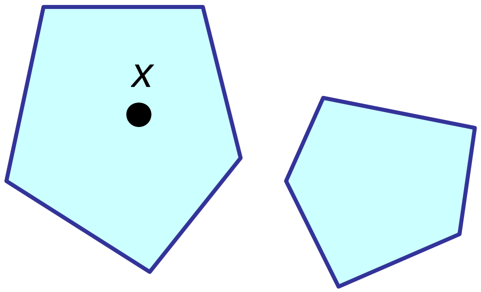



Convex hull formulation

# Idea behind the convex hull formulation

When is this a valid formulation?

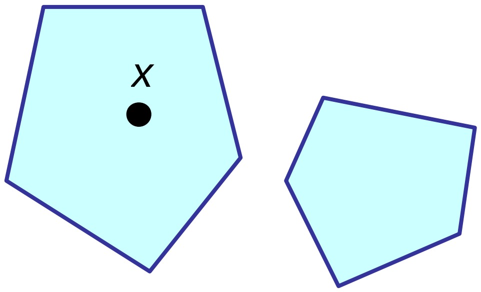
Let's look at an example first…

$$A^k \bar{x}^k \geq b^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$x = \sum_k y_k \bar{x}^k$$

$$\boxed{y_k \in \{0,1\}}$$




Convex hull formulation

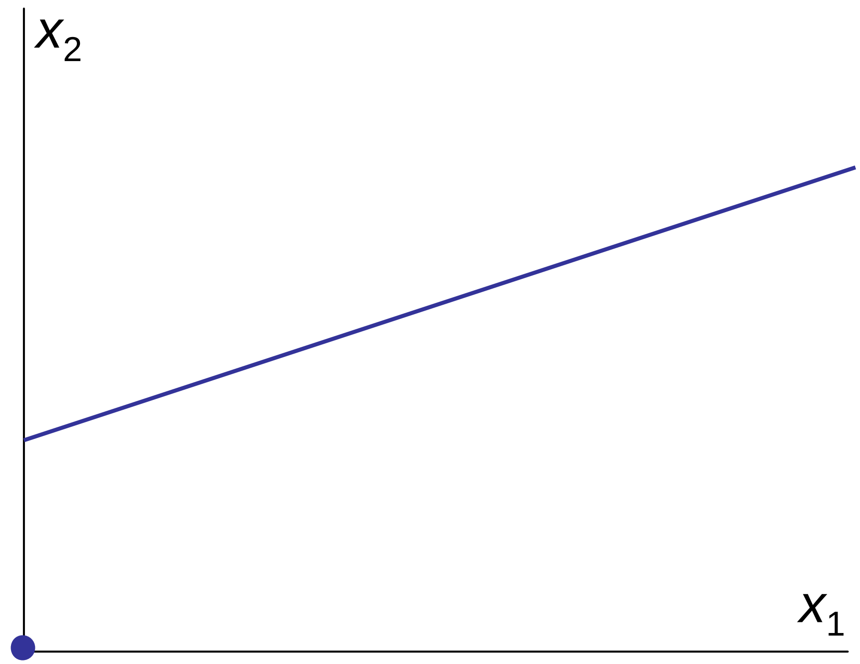
# Example: Fixed charge function

Minimize a fixed charge function:

$$\min \ x_2$$

$$x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases}$$

$$x_1 \geq 0$$



# Fixed charge problem

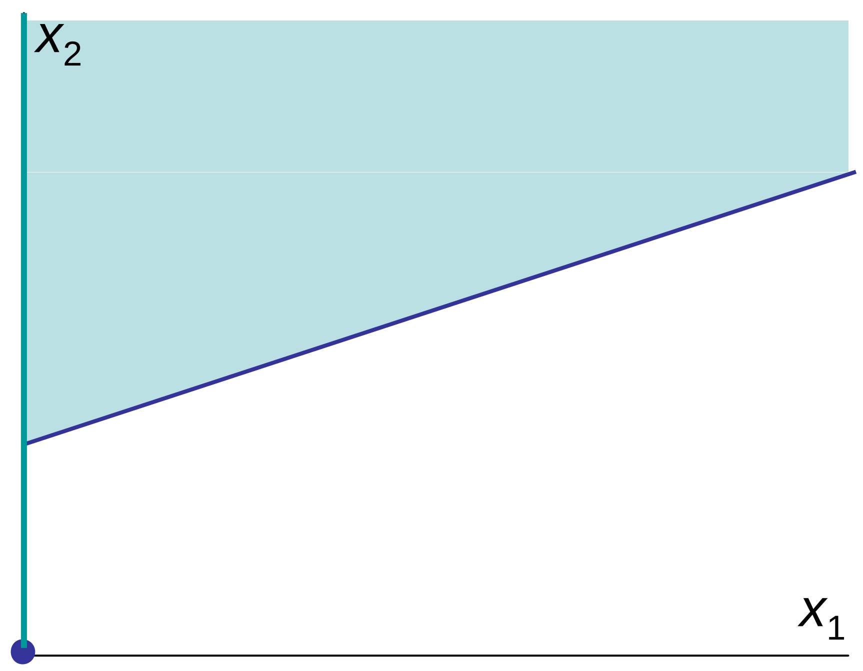
Minimize a fixed charge function:

$$\min \quad x_2$$

$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$x_1 \geq 0$$

Feasible set (epigraph)



# Fixed charge problem

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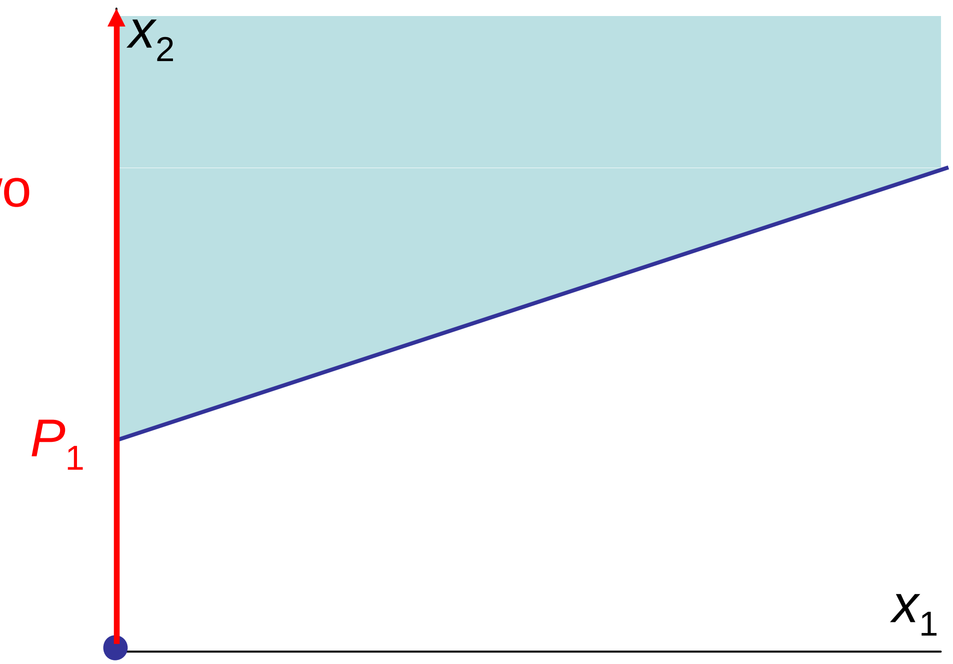
$$x_1 \geq 0$$

Union of two polyhedra $P_1$, $P_2$



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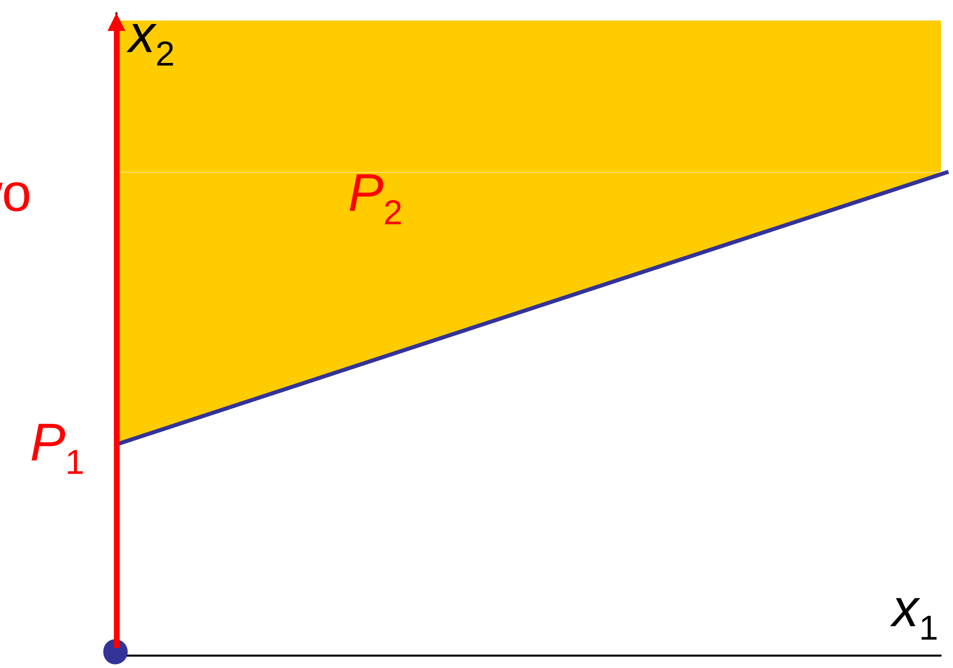
Union of two polyhedra
$P_1$, $P_2$



# Fixed charge problem

Minimize a fixed charge function:

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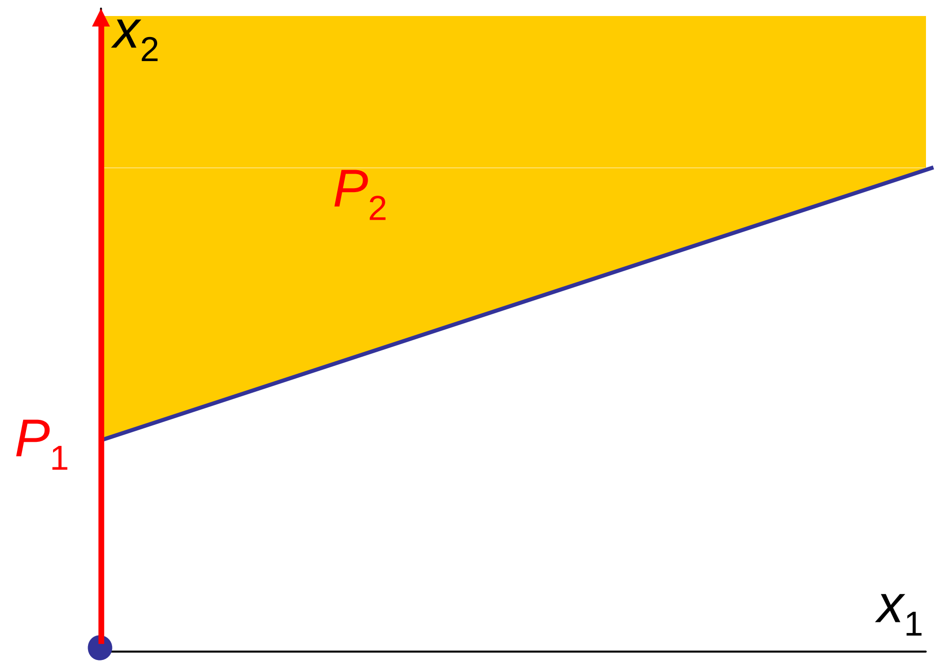
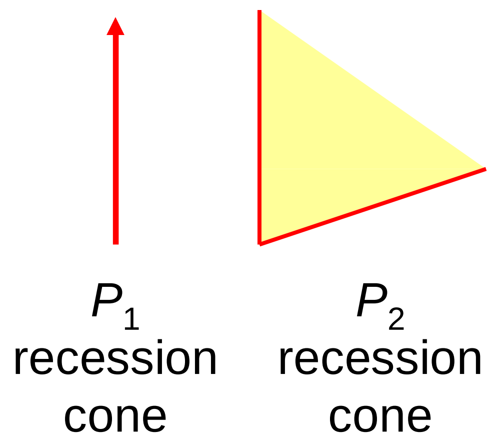
$$x_2 \geq \begin{Bmatrix} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{Bmatrix}$$

$$x_1 \geq 0$$

The polyhedra have different recession cones.



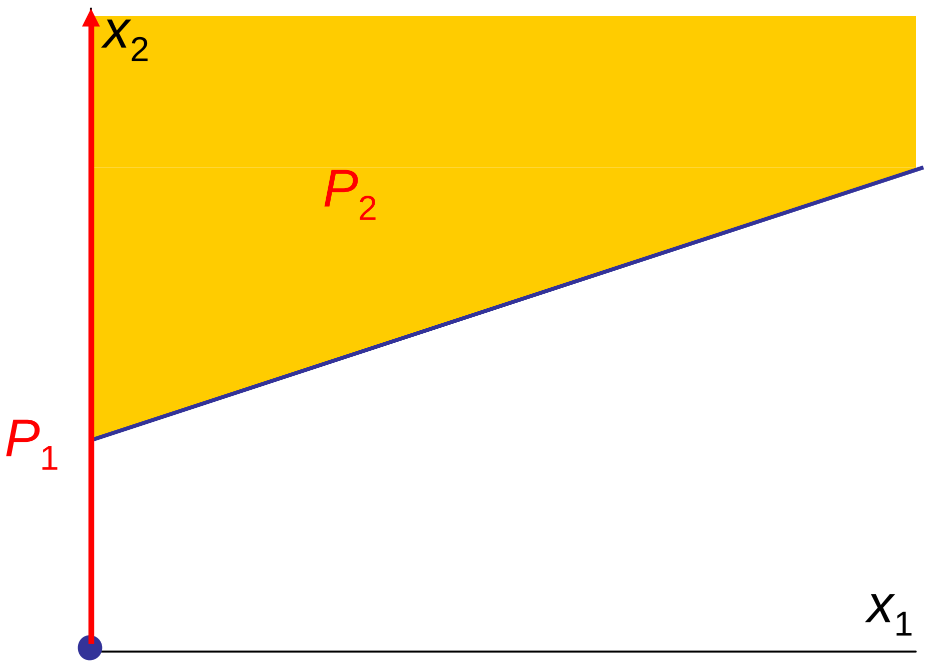



# Fixed charge problem

Disjunctive model describes convex hull relaxation but not the feasible set.

$$\min \ x_2$$

$$\begin{pmatrix} x_1 = 0 \\ x_2 \geq 0 \end{pmatrix} \vee \begin{pmatrix} x_1 \geq 0 \\ x_2 \geq f + cx_1 \end{pmatrix}$$



# Fixed charge problem

Start with a disjunction of linear systems to represent the union of polyhedra

$$\min\ x_2$$

$$\begin{pmatrix} x_1 = 0 \\ x_2 \geq 0 \end{pmatrix} \vee \begin{pmatrix} x_1 \geq 0 \\ x_2 \geq f + cx_1 \end{pmatrix}$$

Introduce a 0-1 variable $y_k$ that is 1 when $x$ is in polyhedron $k$.

Disaggregate $x$ to create an $x^k$ for each $k$.

$$\min\ x_2$$

$$\begin{array}{ll} x_1^1 = 0 & x_1^2 \geq 0 \\ x_2^1 \geq 0 & -cx_1^2 + x_2^2 \geq fy_2 \end{array}$$

$$y_1 + y_2 = 1,\quad y_k \in [0,1]$$

$$x_1 = x_1^1 + x_1^2,\quad x_2 = x_2^1 + x_2^2$$

To simplify, replace $x_1^2$ with $x_1$ since $x_1^1 = 0$

$$\min \ x_2$$

$$x_1^1 = 0 \qquad x_1^2 \geq 0$$

$$x_2^1 \geq 0 \qquad -cx_1^2 + x_2^2 \geq fy_2$$

$$y_1 + y_2 = 1, \ \ y_k \in [0,1]$$

$$x_1 = x_1^1 + x_1^2, \quad x_2 = x_2^1 + x_2^2$$

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$$\min \; x_2$$

$$x_1 \geq 0$$

$$x_2^1 \geq 0 \qquad -cx_1 + x_2^2 \geq fy_2$$

$$y_1 + y_2 = 1, \;\; y_k \in [0,1]$$

$$x_2 = x_2^1 + x_2^2$$

Replace $x_2^2$ with $x_2$

because $x_2^1$ plays no role in the model

$$\min \; x_2$$

$$x_1 \geq 0$$

$$x_2^1 \geq 0 \qquad -cx_1 + x_2^2 \geq fy_2$$

$$y_1 + y_2 = 1, \;\; y_k \in [0,1]$$

$$x_2 = x_2^1 + x_2^2$$

Replace $x_2^2$ with $x_2$

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$$\min \; x_2$$

$$x_1 \geq 0$$

$$-cx_1 + x_2 \geq fy_2$$

$$y_1 + y_2 = 1, \; y_k \in [0,1]$$

Replace $y_2$ with $y$

because $y_1$ plays no role in the model

$$\min \; x_2$$
$$x_1 \geq 0$$
$$-cx_1 + x_2 \geq fy_2$$
$$y_1 + y_2 = 1, \;\; y_k \in [0,1]$$

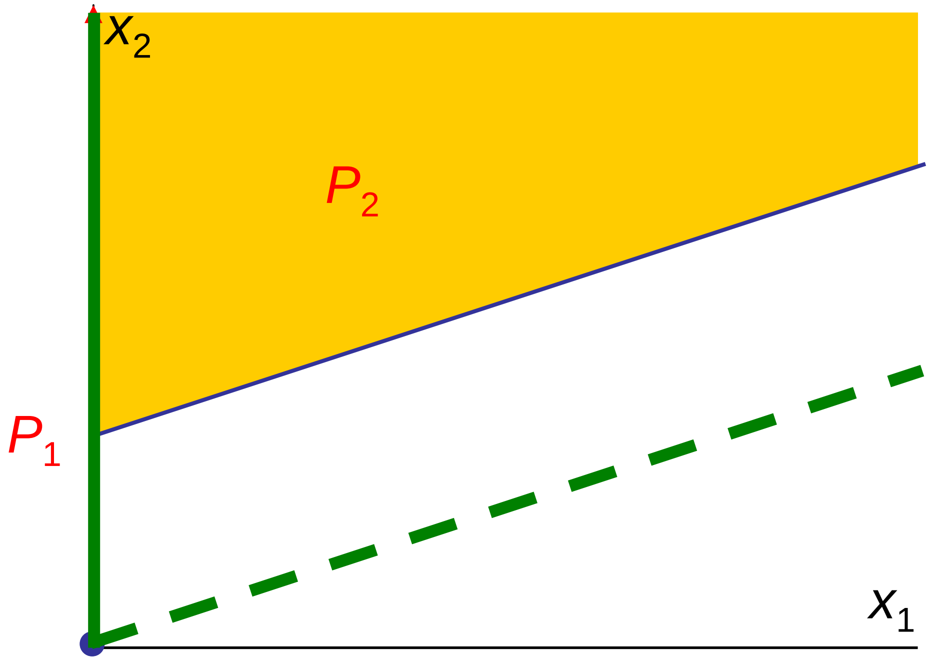
Replace $y_2$ with $y$

because $y_1$ plays no role in the model

$$\min \; x_2$$
$$x_1 \geq 0$$
$$x_2 \geq cx_1 + fy$$
$$y \in [0,1]$$

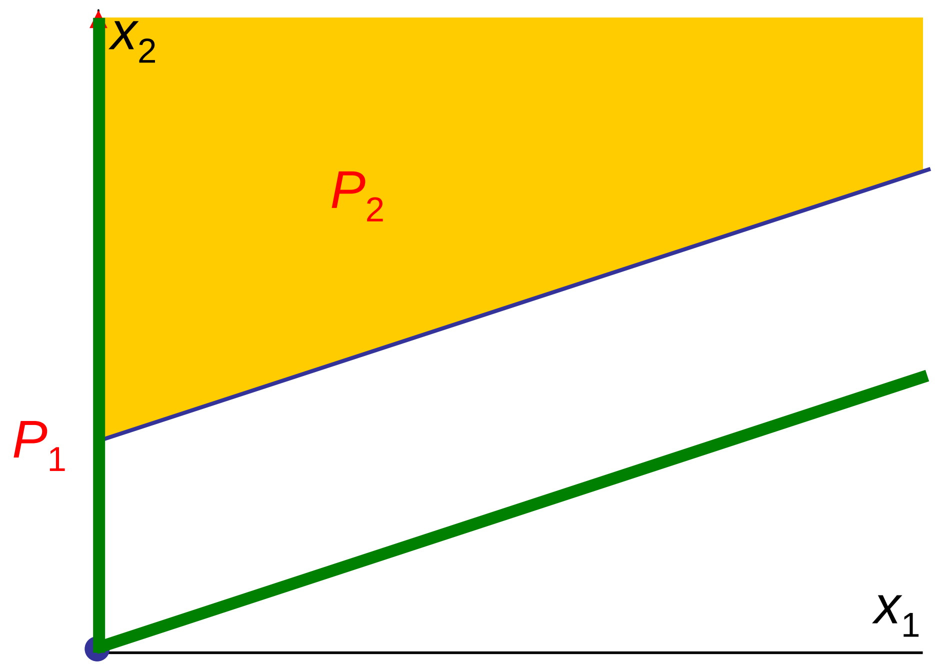
The convex hull is this.

$$\min \quad x_2$$
$$x_1 \geq 0$$
$$x_2 \geq cx_1 + fy$$
$$y \in [0,1]$$



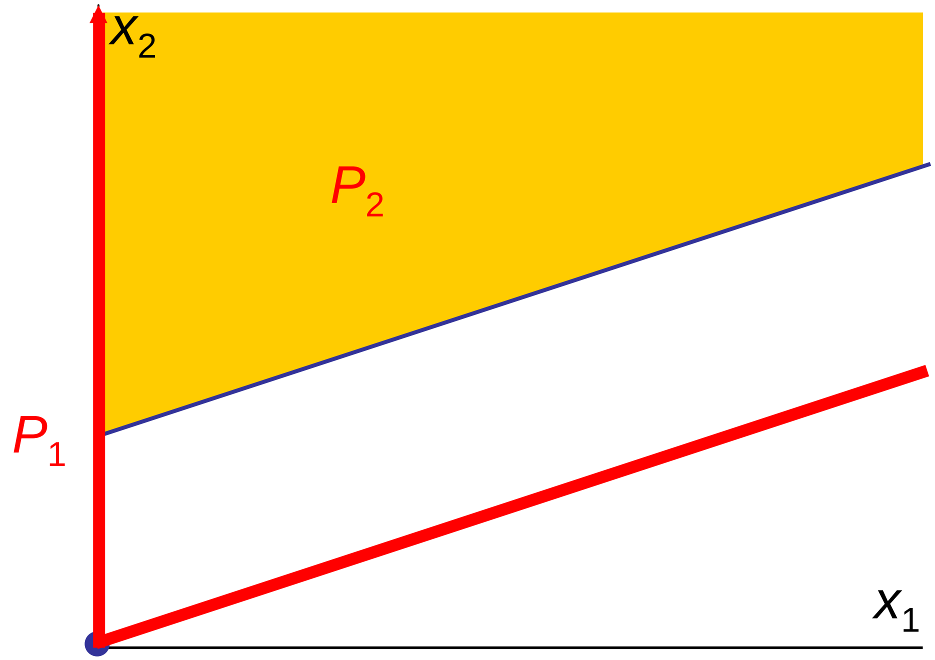
Relaxation correctly describes closure of convex hull

$$\min \; x_2$$
$$x_1 \geq 0$$
$$x_2 \geq cx_1 + fy$$
$$y \in [0,1]$$



But MILP model does not describe feasible set

$$\min \ x_2$$
$$x_1 \geq 0$$
$$x_2 \geq cx_1 + fy$$
$$y \in \{0,1\}$$



## To fix the problem…

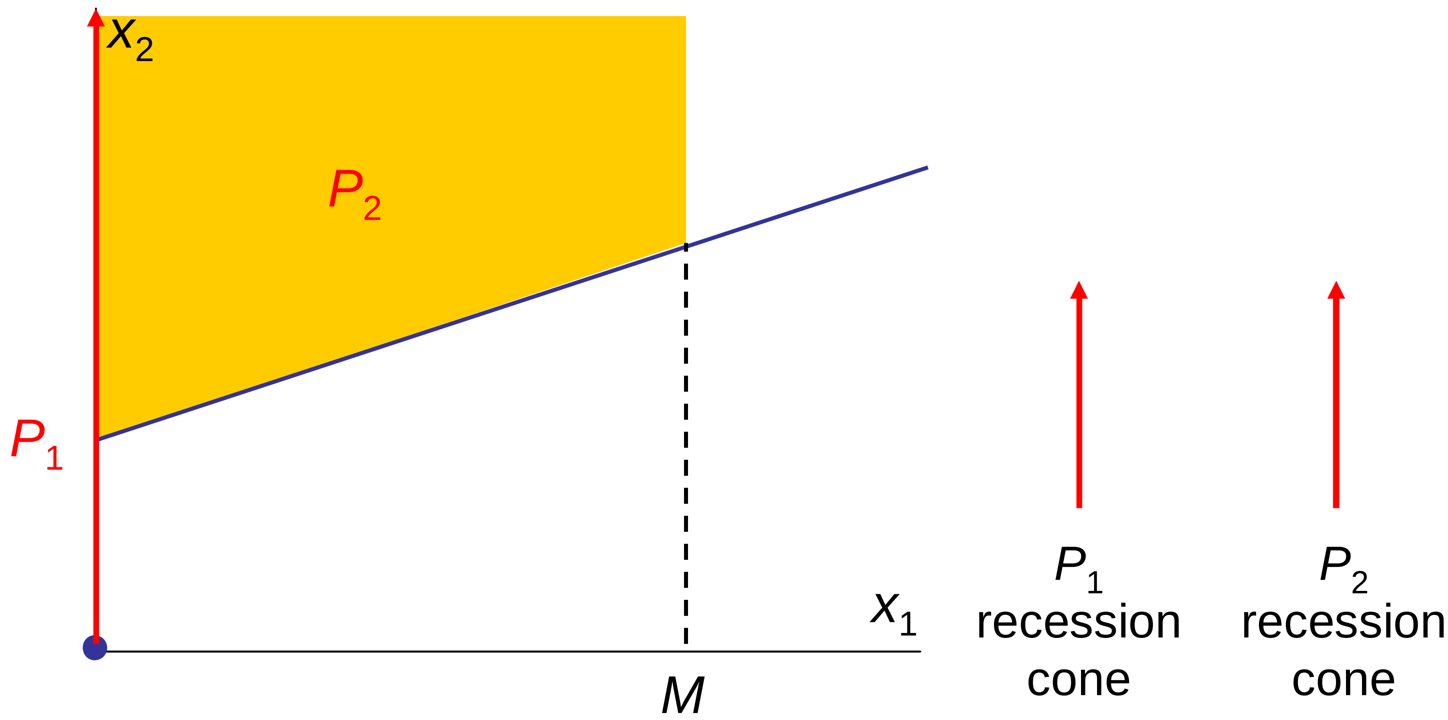
$$\min \ x_2$$

$$x_2 \geq \begin{cases} 0 & \text{if } x_1 = 0 \\ f + cx_1 & \text{if } x_1 > 0 \end{cases}$$

Add an upper bound on $x_1$

$$0 \leq x_1 \leq M$$

The polyhedra have the same recession cone.

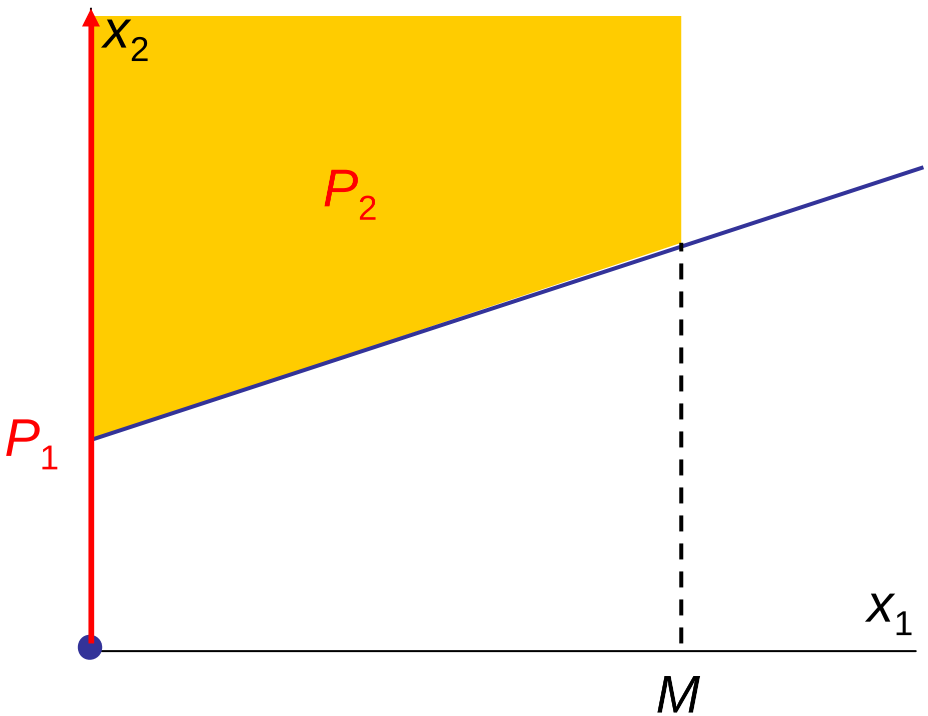


# Fixed charge problem

The disjunction is now…

$$\min \; x_2$$

$$\begin{pmatrix} x_1 = 0 \\ x_2 \geq 0 \end{pmatrix} \vee \begin{pmatrix} 0 \leq x_1 \leq M \\ x_2 \geq f + cx_1 \end{pmatrix}$$



# Fixed charge problem

$$\min \; x_2$$

$$\begin{pmatrix} x_1 = 0 \\ x_2 \ge 0 \end{pmatrix} \vee \begin{pmatrix} 0 \le x_1 \le M \\ x_2 \ge f + cx_1 \end{pmatrix}$$

The disjunctive model is

$$\begin{aligned} &\min \; x_2 \\ &x_1^1 = 0 \qquad\qquad 0 \le x_1^2 \le My_2 \\ &x_2^1 \ge 0 \qquad -cx_1^2 + x_2^2 \ge fy_2 \\ &y_1 + y_2 = 1, \;\; y_k \in \{0,1\} \\ &x_1 = x_1^1 + x_1^2, \quad x_2 = x_2^1 + x_2^2 \end{aligned}$$

This simplifies as before…

$$\min \; x_2$$

$$0 \le x_1 \le My$$

$$x_2 \ge cx_1 + fy$$

$$y \in \{0,1\}$$

This simplifies as before…

$$\begin{aligned}
&\min\ x_2 \\
&0 \le x_1 \le My \\
&x_2 \ge cx_1 + fy \\
&y \in \{0,1\}
\end{aligned}$$

Previous model

$$\begin{aligned}
&\min\ x_2 \\
&\quad x_1 \ge 0 \\
&x_2 \ge cx_1 + fy \\
&y \in \{0,1\}
\end{aligned}$$

This simplifies as before…

$$\min\ x_2$$
$$0 \le x_1 \le My$$
$$x_2 \ge cx_1 + fy$$
$$y \in \{0,1\}$$

or

$$\min\ cx + fy$$
$$0 \le x \le My$$
$$y \in \{0,1\}$$

"Big *M*"

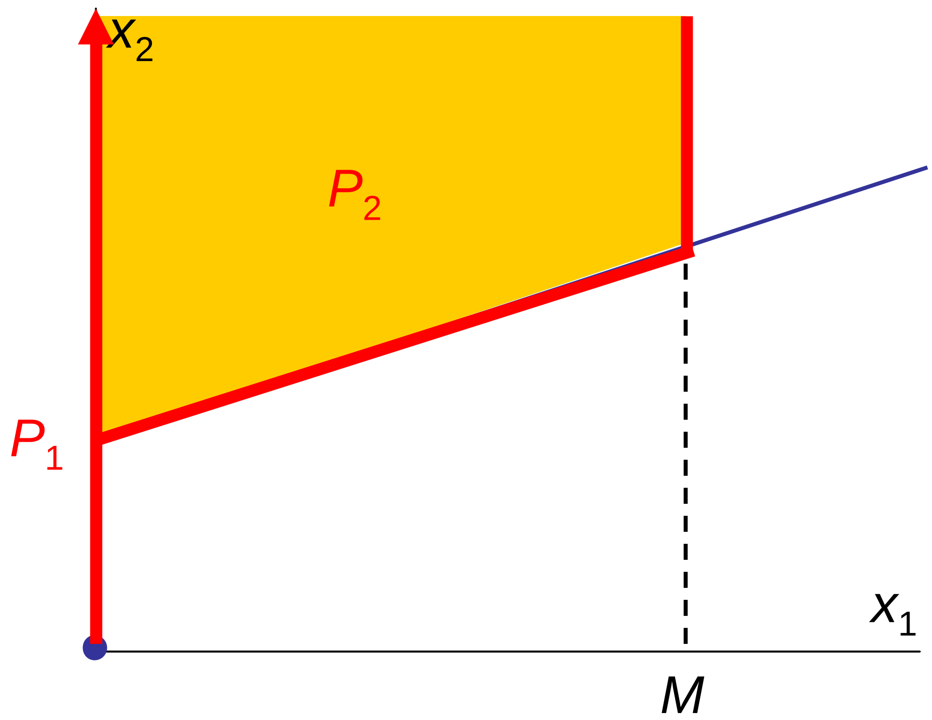
Previous model

$$\min\ x_2$$
$$x_1 \ge 0$$
$$x_2 \ge cx_1 + fy$$
$$y \in \{0,1\}$$

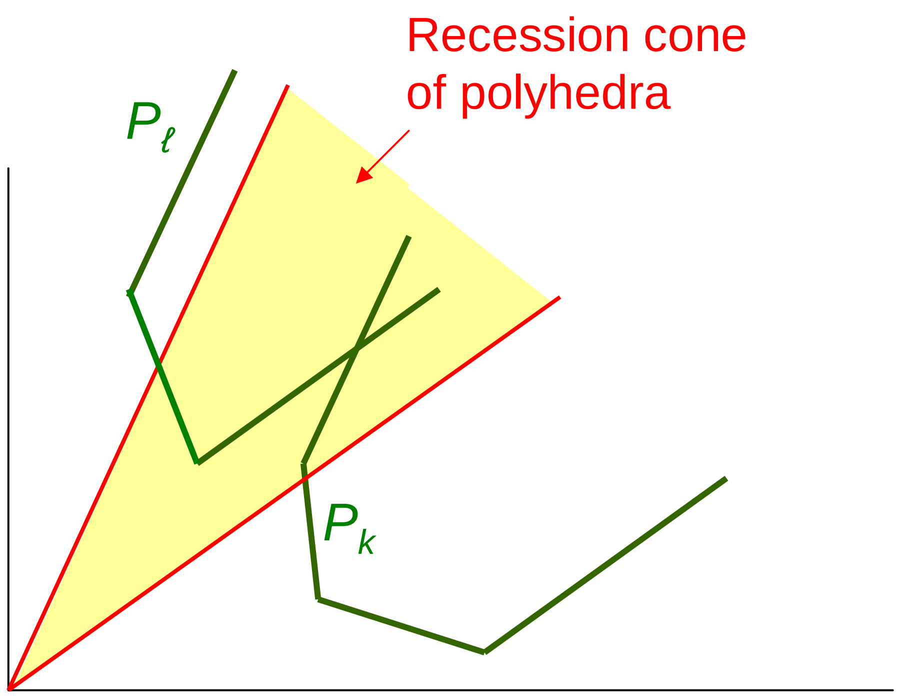
The model now correctly describes the feasible set.

$$\min \; cx + fy$$
$$0 \le x \le My$$
$$y \in \{0,1\}$$

"Big $M$"



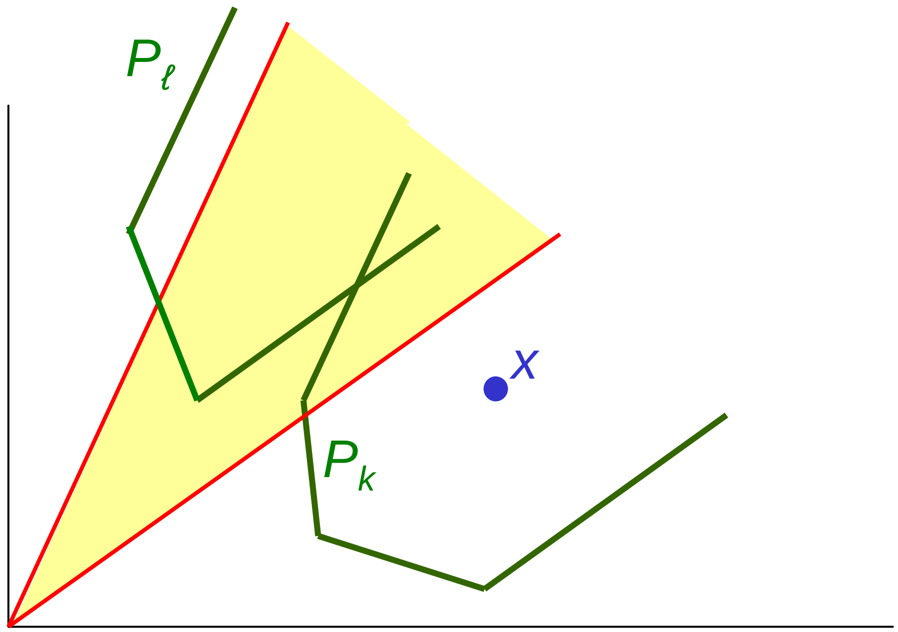
# Why the disjunctive model works




$$\min \; cx$$
$$A^k x^k \ge b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in \{0,1\}$$

Let S be feasible set.

# Why the disjunctive model works

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Let S be feasible set.

$x \in S \quad \Rightarrow \quad x \in$ some $P_k$

## Why the disjunctive model works

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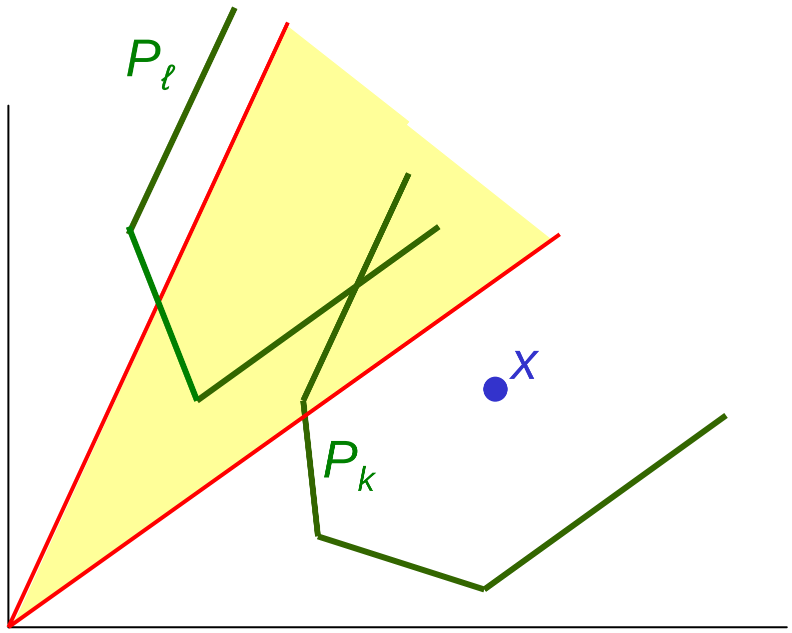



Let S be feasible set.

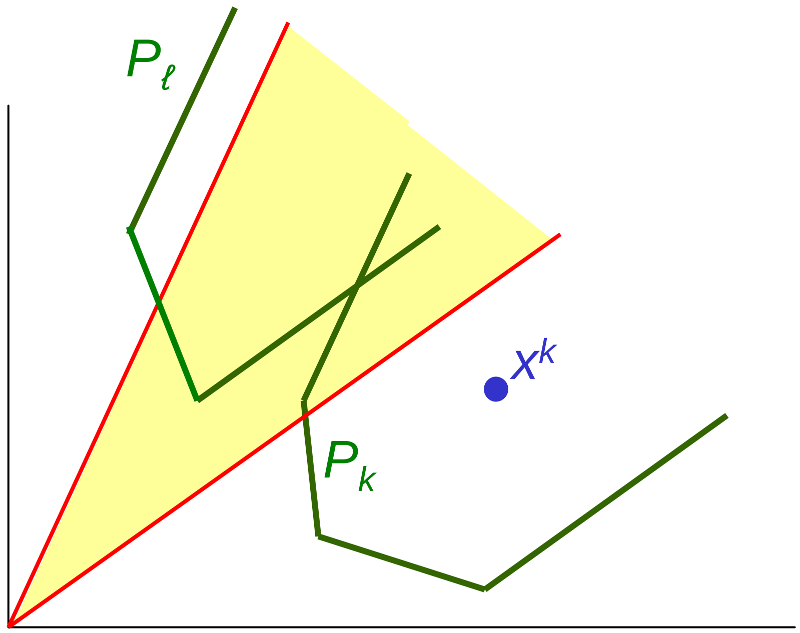
$x \in S \quad \Rightarrow \quad x \in$ some $P_k$

$\Rightarrow \quad x$ satisfies the model for

$y_k = 1$, other $y_\ell s = 0$

$x^k = x$, other $x^\ell s = 0$

# Why the disjunctive model works



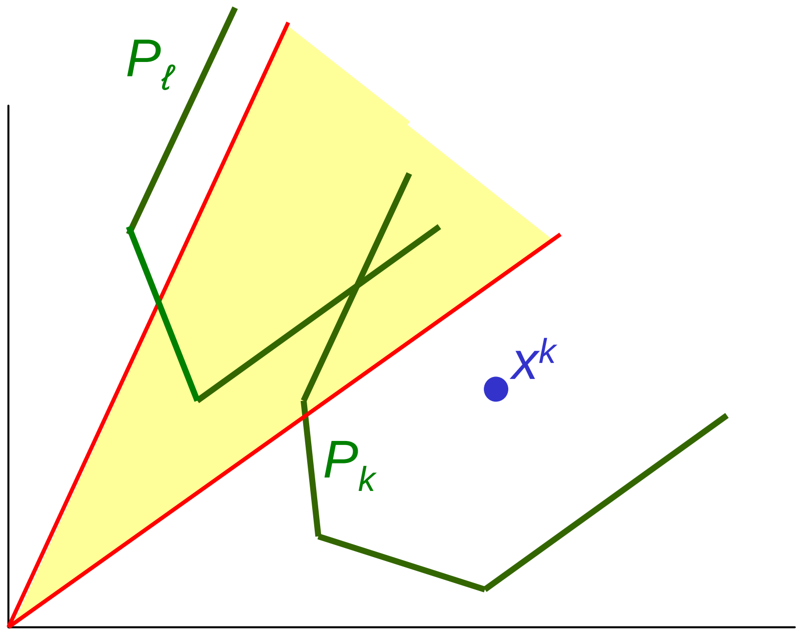

$$\min \; cx$$
$$A^k x^k \ge b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in \{0,1\}$$

Conversely, suppose

$x$, $y$, $x^k$s satisfy the model

$\Rightarrow$ some $y_k = 1$ $\Rightarrow$ $x^k \in P_k$

# Why the disjunctive model works




$$\min\ cx$$
$$A^k x^k \geq b^k y_k,\ \text{all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in \{0,1\}$$

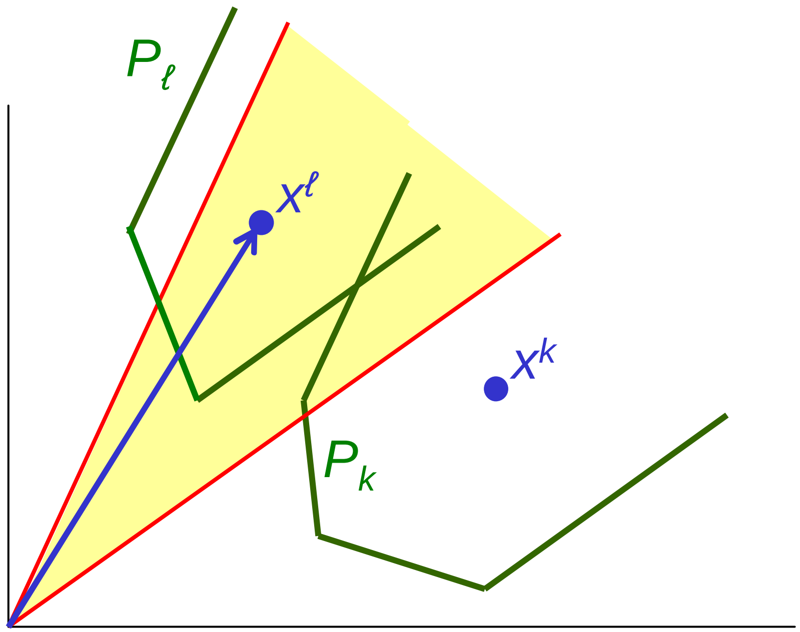
Conversely, suppose

$x$, $y$, $x^k$s satisfy the model

$\Rightarrow$ some $y_k = 1$ $\Rightarrow$ $x^k \in P_k$

$\Rightarrow$ $A^\ell x^\ell \geq 0$ for other $\ell$s

## Why the disjunctive model works




$$\min\ cx$$
$$A^k x^k \geq b^k y_k,\ \text{all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in \{0,1\}$$

Conversely, suppose

$x$, $y$, $x^k$s satisfy the model

$\Rightarrow$ some $y_k = 1$ $\Rightarrow$ $x^k \in P_k$

$\Rightarrow$ $A^\ell x^\ell \geq 0$ for other $\ell$s

$\Rightarrow$ $x^\ell$s are recession directions for other $P_\ell$s

# Why the disjunctive model works

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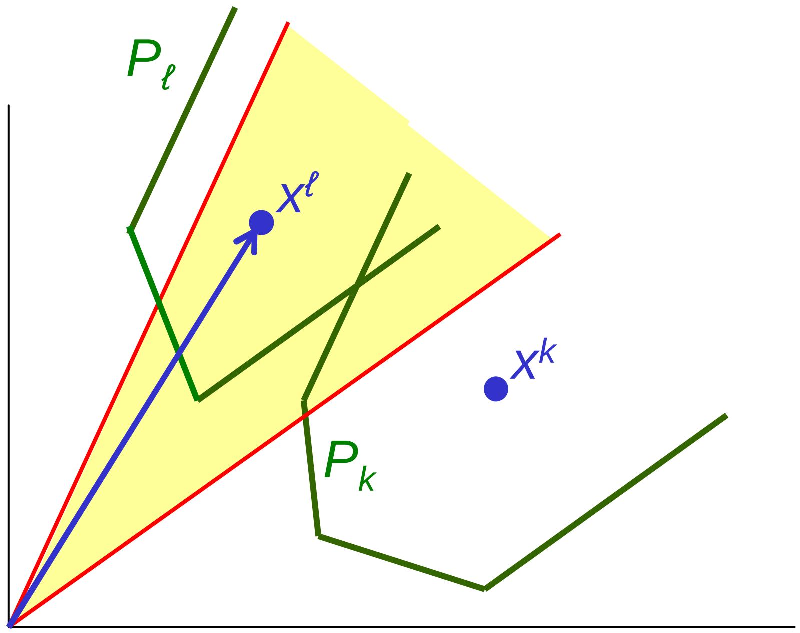


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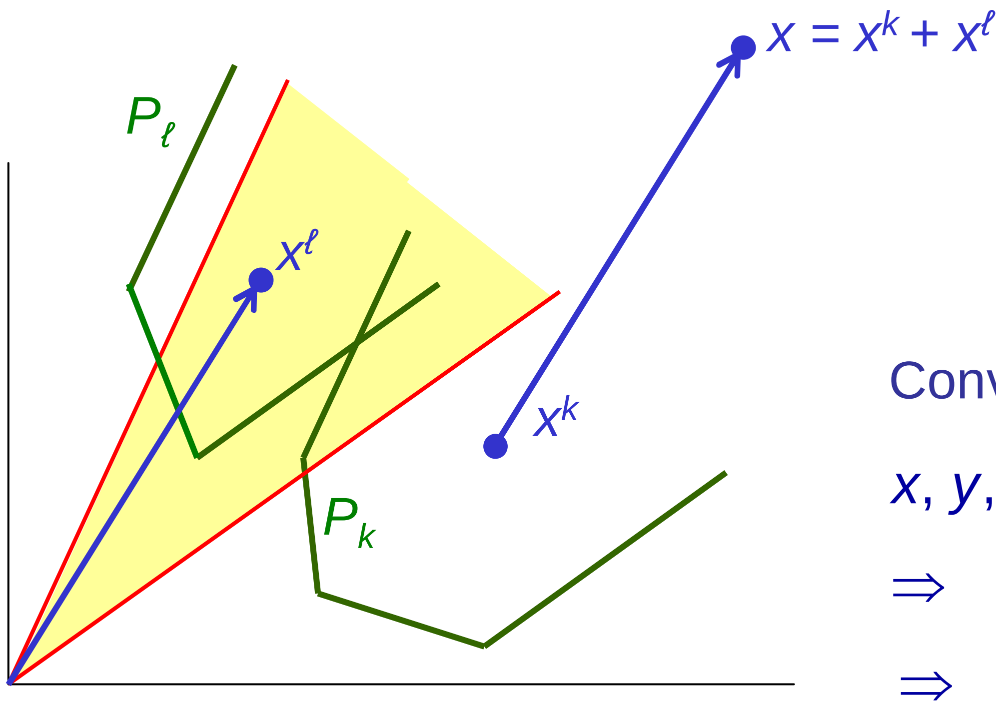


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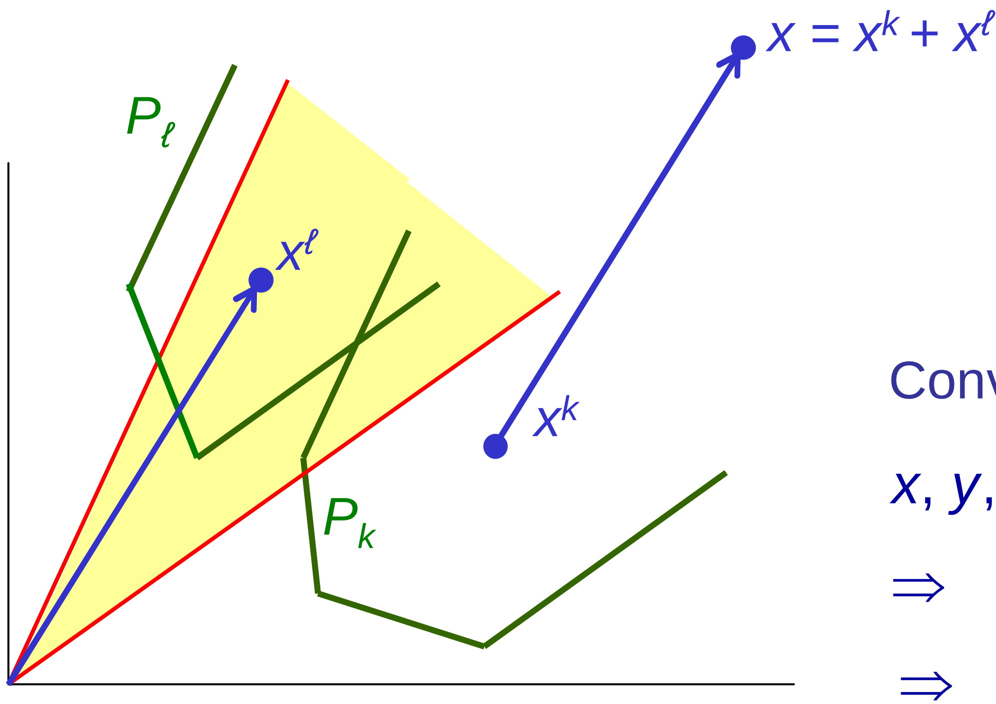
$$\Rightarrow\quad A^k x^\ell \geq 0 \quad\Rightarrow\quad A^k x = A^k\left(x^k + \sum_{\ell \neq k} x^\ell\right) \geq b^k$$

# Why the disjunctive model works

$$\min \; cx$$
$$A^k x^k \geq b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$y_k \in \{0,1\}$$

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$$\Rightarrow \; A^k x^\ell \geq 0 \;\Rightarrow\; A^k x = A^k \left( x^k + \sum_{\ell \neq k} x^\ell \right) \geq b^k$$

$$\Rightarrow \; x \in P_k \;\Rightarrow\; x \in S$$

# Multiple disjunctions

Combining individual convex hull formulations for two disjunctions…

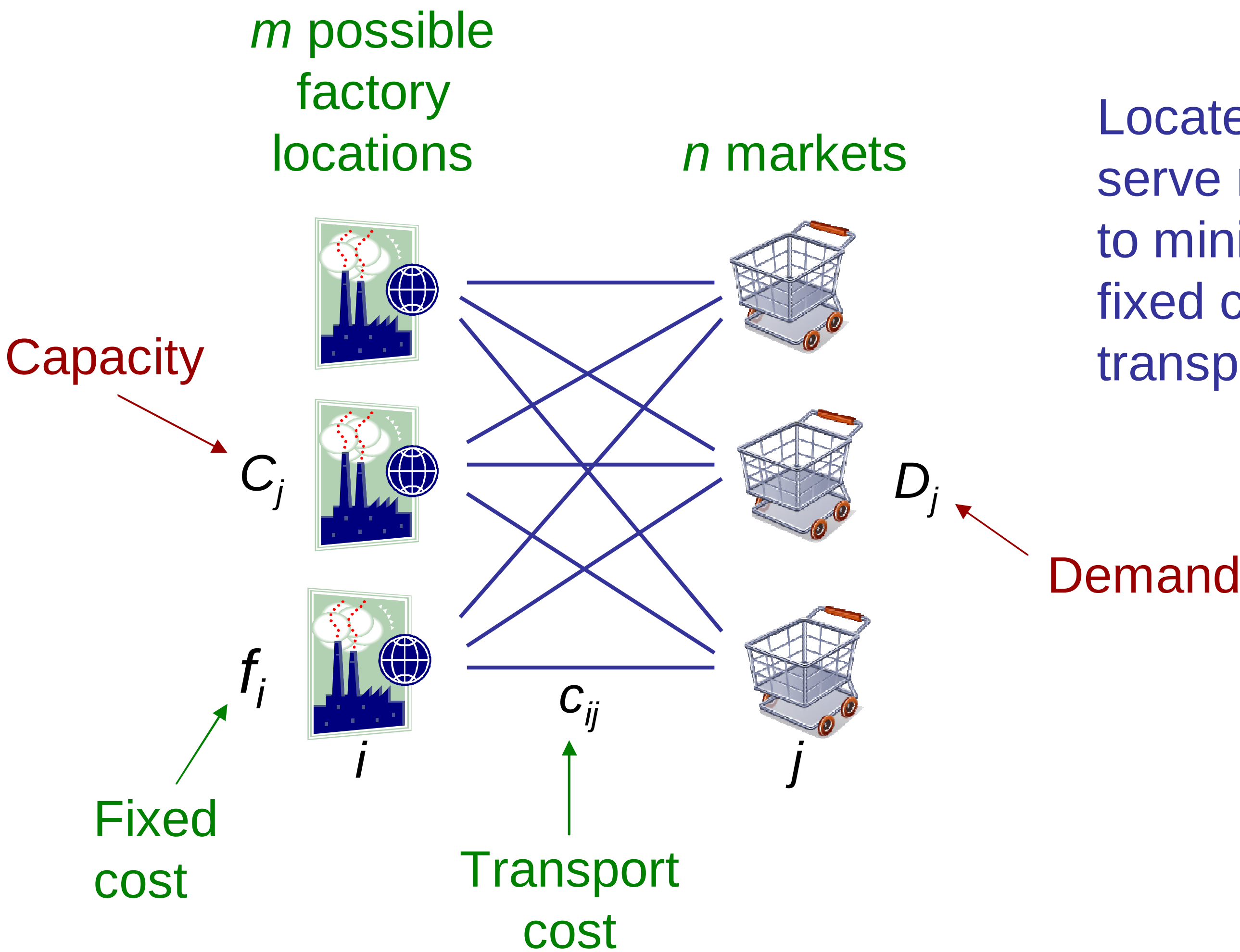
$$\bigvee_k \left( A^k x \geq a^k \right)$$

$$\bigvee_k \left( B^k x \geq b^k \right)$$

does not necessarily produce a convex hull formulation for the pair…

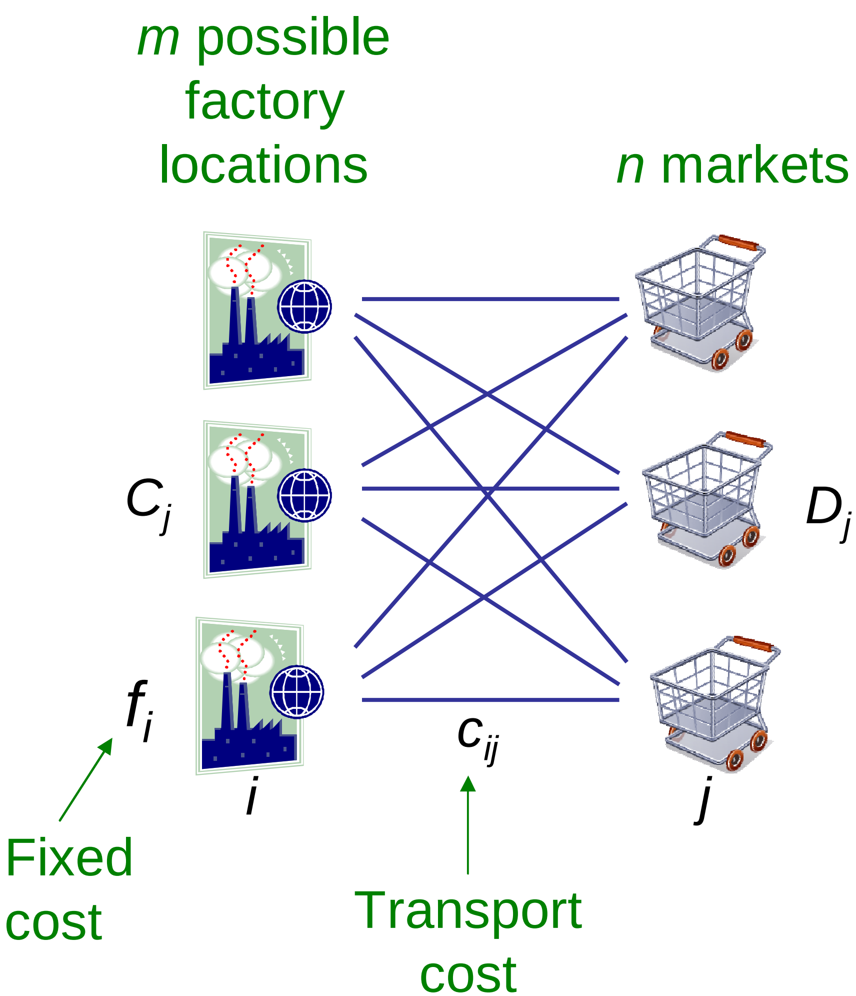
**Theorem.** …unless the disjunctions have no common variables.

# Example:  Facility location




Locate factories to serve markets so as to minimize total fixed cost and transport cost.

# Facility location




Disjunctive model:

Amount shipped from factory *i* to market *j*: $x_{ij}$

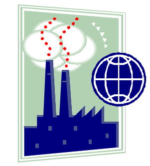
$$\min \sum_i z_i + \sum_{ij} c_{ij} x_{ij}$$

$$\begin{pmatrix} \sum_j x_{ij} \le C_i \\ z_i \ge f_i \\ x_{ij} \ge 0, \text{ all j} \end{pmatrix} \vee \begin{pmatrix} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{pmatrix}, \quad \text{all } i$$

$$\sum_i x_{ij} = D_j, \quad \text{all } j$$

Factory at location *i*

No factory at location *i*



# Facility location

**Disjunctive model:**

$$\min \sum_i z_i + \sum_{ij} c_{ij} x_{ij}$$

$$\begin{pmatrix} \sum_j x_{ij} \le C_i \\ z_i \ge f_i \\ x_{ij} \ge 0, \text{ all j} \end{pmatrix} \vee \begin{pmatrix} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{pmatrix}, \quad \text{all } i$$

$$\sum_i x_{ij} = D_j, \quad \text{all } j$$

**MILP formulation:**

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$

$$\sum_j x_{ij} \le C_i y_i, \text{ all } i$$

$$\sum_i x_{ij} = D_j, \text{ all } j$$

$$y_i \in \{0,1\}, \quad x_{ij} \ge 0, \text{ all } i, j$$

# Uncapacitated facility location

Beginner's mistake:  Model it as special case of capacitated problem

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} \boxed{x_{ij}}$$

$$\sum_j x_{ij} \le \boxed{n y_i}, \ \text{all } i$$

$$y_i \in \{0,1\}$$

$$\sum_i x_{ij} = 1, \ \text{all } j$$

Fraction of demand *j* satisfied by factory *i*

Factory *i* has max output *n*

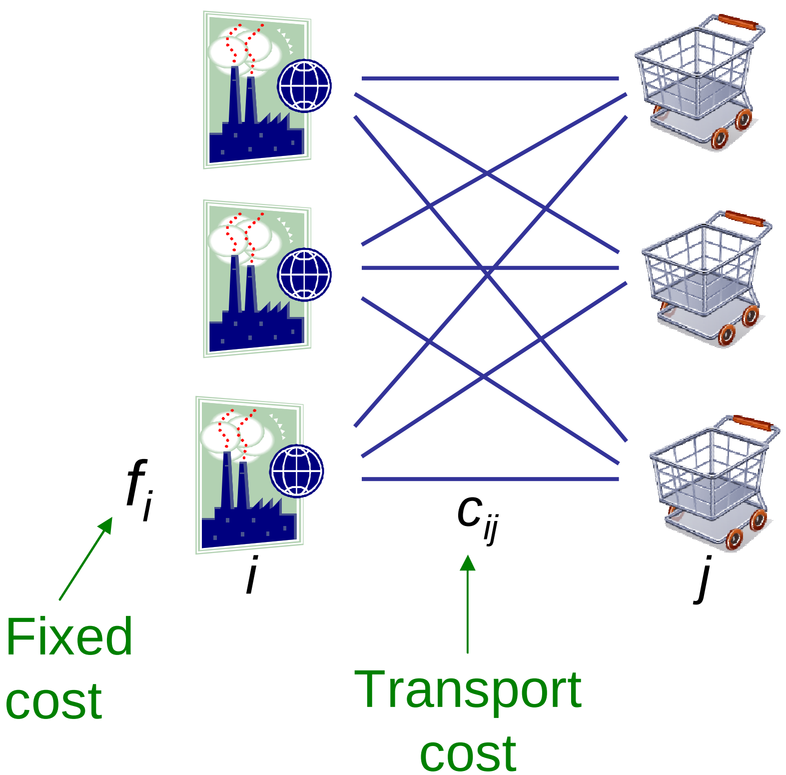
This is not the best model.

We can obtain a tighter model by starting with disjunctive formulation.

# Uncapacitated facility location

*m* possible factory locations

*n* markets




Disjunctive model:

Fraction of demand *j* satisfied by factory *i*

$$\min \sum_i z_i + \sum_{ij} c_{ij} x_{ij}$$

$$\begin{pmatrix} 0 \le x_{ij} \le 1, \text{ all } j \\ z_i \ge f_i \end{pmatrix} \vee \begin{pmatrix} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{pmatrix}, \text{ all } i$$

$$\sum_i x_{ij} = 1, \text{ all } j$$

Factory at location *i*

No factory at location *i*

# Uncapacitated facility location

MILP formulation:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$
$$0 \le x_{ij} \le y_i, \text{ all } i, j$$
$$\sum_i x_{ij} = 1, \text{ all } j$$
$$y_i \in \{0,1\}, \text{ all } i$$

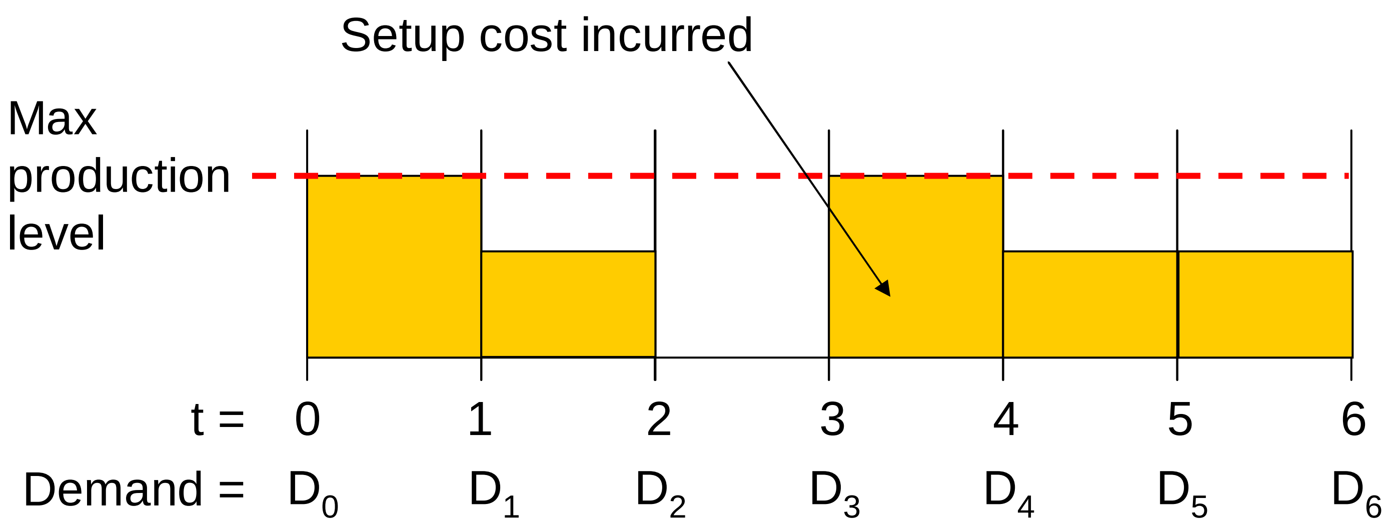
This is the textbook model.

More constraints, but tighter relaxation.

Beginner's model:

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} x_{ij}$$
$$\sum_j x_{ij} \le n y_i, \text{ all } i$$
$$\sum_i x_{ij} = 1, \text{ all } j$$
$$y_i \in \{0,1\}, \text{ all } i$$

# Example: Lot sizing with setup costs




Determine lot size in each period to minimize total production, inventory, and setup costs.

Fixed-cost variable — Fixed cost — Production capacity — Production level

$$\begin{pmatrix} v_t \ge f_t \\ 0 \le x_t \le C_t \end{pmatrix} \vee \begin{pmatrix} v_t \ge 0 \\ 0 \le x_t \le C_t \end{pmatrix} \vee \begin{pmatrix} v_t \ge 0_t \\ x_t = 0 \end{pmatrix}$$

(1) Start production (incurs setup cost)

(2) Continue production (no setup cost)

(3) Produce nothing (no production cost)

Logical conditions:

(2) In period $t$ $\Rightarrow$ (1) or (2) in period $t - 1$

(1) In period $t$ $\Rightarrow$ neither (1) nor (2) in period $t - 1$

| (1) | (2) | (3) |
|---|---|---|
| Start production | Continue production | Produce nothing |

$$\begin{pmatrix} v_t \ge f_t \\ 0 \le x_t \le C_t \end{pmatrix} \vee \begin{pmatrix} v_t \ge 0 \\ 0 \le x_t \le C_t \end{pmatrix} \vee \begin{pmatrix} v_t \ge 0_t \\ x_t = 0 \end{pmatrix}$$

Convex hull MILP model of disjunction:

$$\begin{matrix} v_t^1 \ge f_t y_{t1} & v_t^2 \ge 0 & v_t^3 \ge 0 \\ 0 \le x_t^1 \le C_t y_{t1} & 0 \le x_t^2 \le C_t y_{t2} & x_t^3 = 0 \end{matrix}$$

$$v_t = \sum_{k=1}^{3} v_t^k, \quad x_t = \sum_{k=1}^{3} x_t^k, \quad y_t = \sum_{k=1}^{3} y_{tk}$$

$$y_{tk} \in \{0,1\}, \quad k = 1,2,3$$

To simplify, define

$$z_t = y_{t1}$$

$$y_t = y_{t2}$$

Convex hull MILP model of disjunction:

$$v_t^1 \geq f_t y_{t1} \qquad v_t^2 \geq 0 \qquad v_t^3 \geq 0$$

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$$v_t = \sum_{k=1}^{3} v_t^k, \quad x_t = \sum_{k=1}^{3} x_t^k, \quad z_t + y_t \le 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

= 1 for startup ($z_t$)

= 1 for continued production ($y_t$)

Since $x_t^3 = 0$
set $x_t = x_t^1 + x_t^2$

Convex hull MILP model of disjunction:

$$v_t^1 \geq f_t z_t \qquad v_t^2 \geq 0 \qquad v_t^3 \geq 0$$

$$0 \leq x_t^1 \leq C_t z_t \qquad 0 \leq x_t^2 \leq C_t y_t \qquad x_t^3 = 0$$

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Since $x_t^3 = 0$
set $x_1 = x_1^1 + x_2^2$

Convex hull MILP model of disjunction:

$$v_t^1 \geq f_t z_t \qquad v_t^2 \geq 0 \qquad v_t^3 \geq 0$$

$$0 \leq x_t \leq C_t(z_t + y_t)$$

$$v_t = \sum_{k=1}^{3} v_t^k, \quad z_t + y_t \leq 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

Since $v_t$ occurs positively in the objective function, and $v_t^2, v_t^3$ do not play a role, let $v_t = v_t^1$

Convex hull MILP model of disjunction:

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Convex hull MILP model of disjunction:

$$
\begin{aligned}
& v_t \geq f_t z_t \\
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& z_t, y_t \in \{0,1\}, \quad k = 1,2,3
\end{aligned}
$$

Formulate logical conditions:

(2) In period $t$ $\Rightarrow$ (1) or (2) in period $t-1$

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$$v_t \geq f_t z_t$$
$$0 \leq x_t \leq C_t(z_t + y_t)$$
$$z_t + y_t \leq 1$$
$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$
$$y_t \leq z_{t-1} + y_{t-1}$$
$$z_t \leq 1 - z_{t-1} - y_{t-1}$$

Add objective function

Unit production cost

Unit holding cost

$$\min \quad \sum_{t=1}^{n} (p_t x_t + h_t s_t + v_t)$$

$$v_t \geq f_t z_t$$

$$0 \leq x_t \leq C_t (z_t + y_t)$$

$$z_t + y_t \leq 1$$

$$z_t, y_t \in \{0,1\}, \quad k = 1,2,3$$

$$y_t \leq z_{t-1} + y_{t-1}$$

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# Logical variables

To tighten an MILP formulation of

$$A \vee B \vee C \vee D$$
$$E \vee F \vee G$$
$$(y_A \wedge y_B) \rightarrow y_E$$

Put logical constraint in CNF:

$$\neg y_A \vee \neg y_B \vee y_E$$

Replace negative with positive variables: $C \vee D \vee E$

And add convex hull formulation of this clause.

**Conjecture:** this does not tighten the formulation when the disjunctions have no variables in common.

# Big-M Disjunctive Formulation

Again start with a disjunction of linear systems.

$$\bigvee_k \left( A^k x \geq b^k \right)$$

Big $M$

$y_k$ is 1 when $x$ is in polyhedron $k$.

$$A^k x^k \geq b^k - (1 - y_k) M^k, \text{ all } k$$

$$\sum_k y_k = 1$$

$$y_k \in \{0,1\}, \text{ all } k$$

$M^k$ is a vector of bounds that makes system $k$ nonbinding when $y_k = 0$.

$$M^k = b^k - \min \left\{ A^k x \,\middle|\, \bigvee_{\ell \neq k} \left( A^\ell x \geq b^\ell \right) \right\}$$

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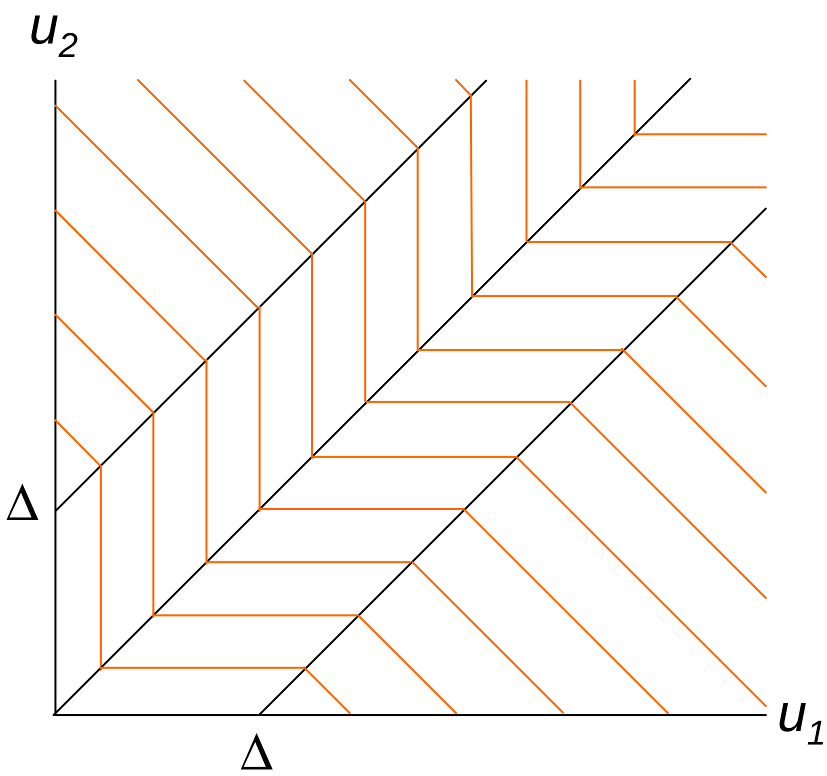
Every bounded MILP-representable set has a model of this form (as well as a convex hull disjunctive model).

# Example: Health Care Benefits

Distribute limited health benefits to two persons.
Person $i$ receives utility $u_i$.




Two criteria:

If $|u_1 - u_2| \leq \Delta$, Rawlsian:

$\max \min\{u_1, u_2\}$

If $|u_1 - u_2| > \Delta$, utilitarian:

$\max u_1 + u_2$

Maximize welfare of person who is more seriously ill, unless this requires too much sacrifice from the other person.

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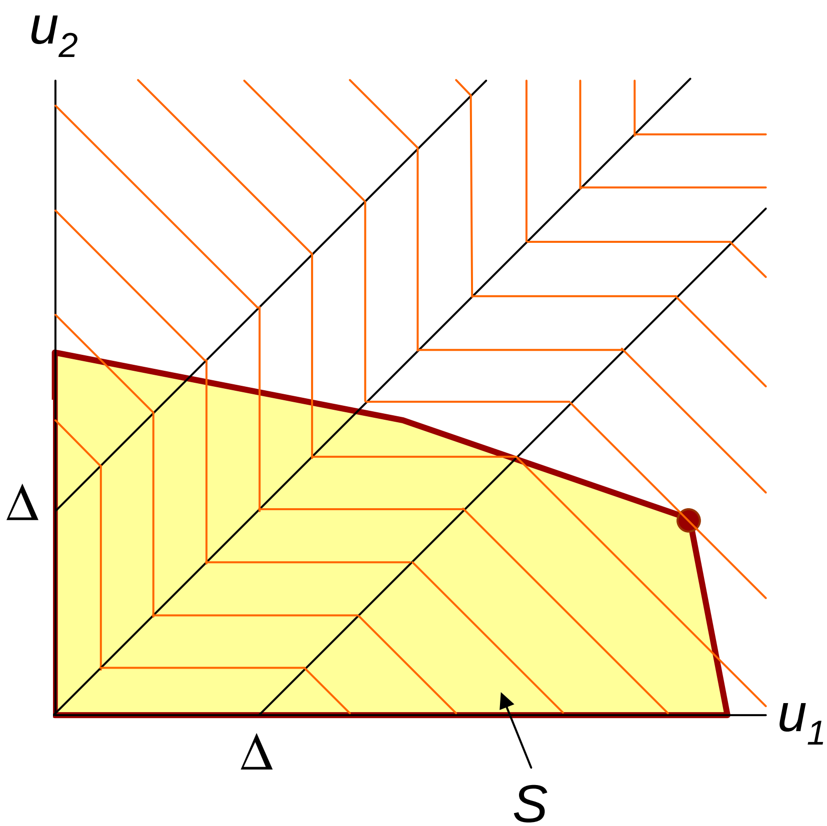


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Optimization problem:

$$\max z$$

$$z \leq \begin{cases} 2\min\{u_1, u_2\} + \Delta & \text{if } |u_1 - u_2| \leq \Delta \\ u_1 + u_2 & \text{otherwise} \end{cases}$$

$$u_1, u_2 \in S$$

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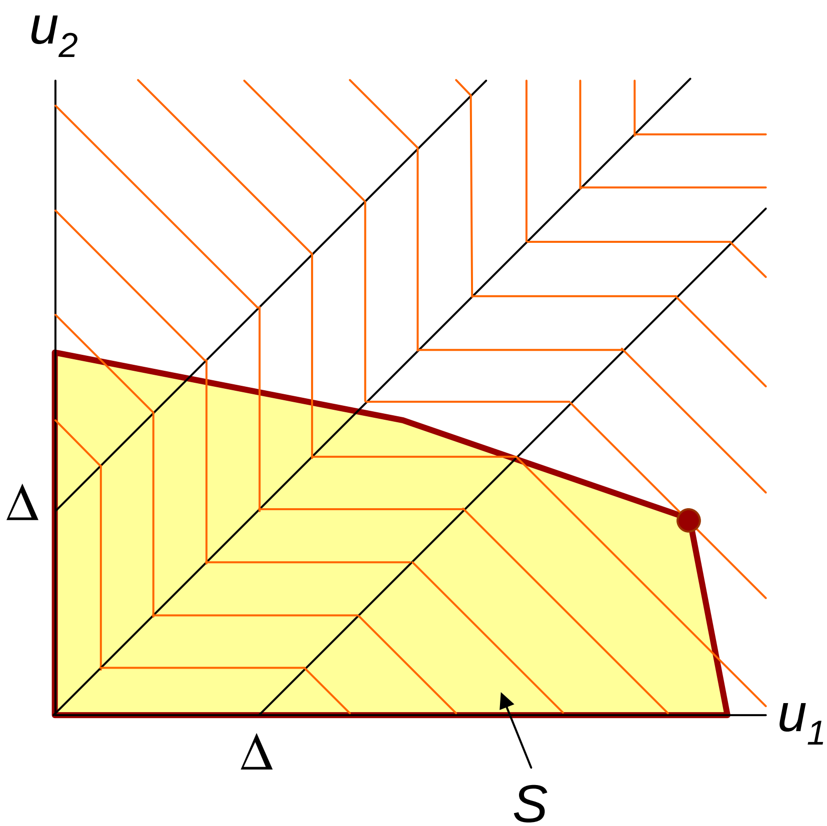


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$$u_1, u_2 \in S$$

Ensures continuity

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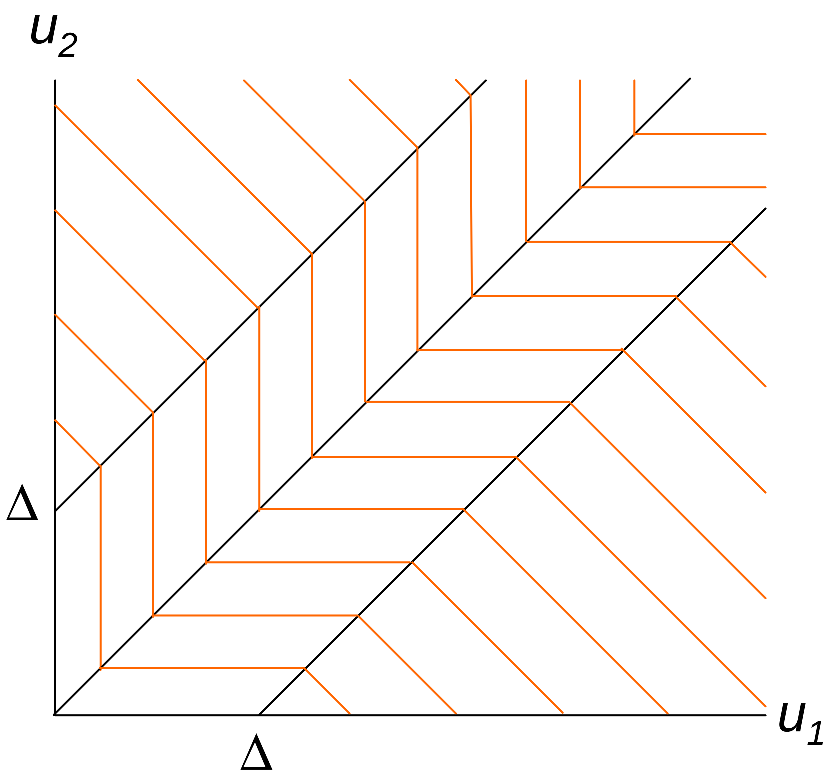


Ignoring $S$, we would like a convex hull MILP model of the epigraph.

Can we do it?

No!

Optimization problem:

$$\max z$$

$$z \le \begin{cases} 2\min\{u_1, u_2\} + \Delta & \text{if } |u_1 - u_2| \le \Delta \\ u_1 + u_2 & \text{otherwise} \end{cases}$$

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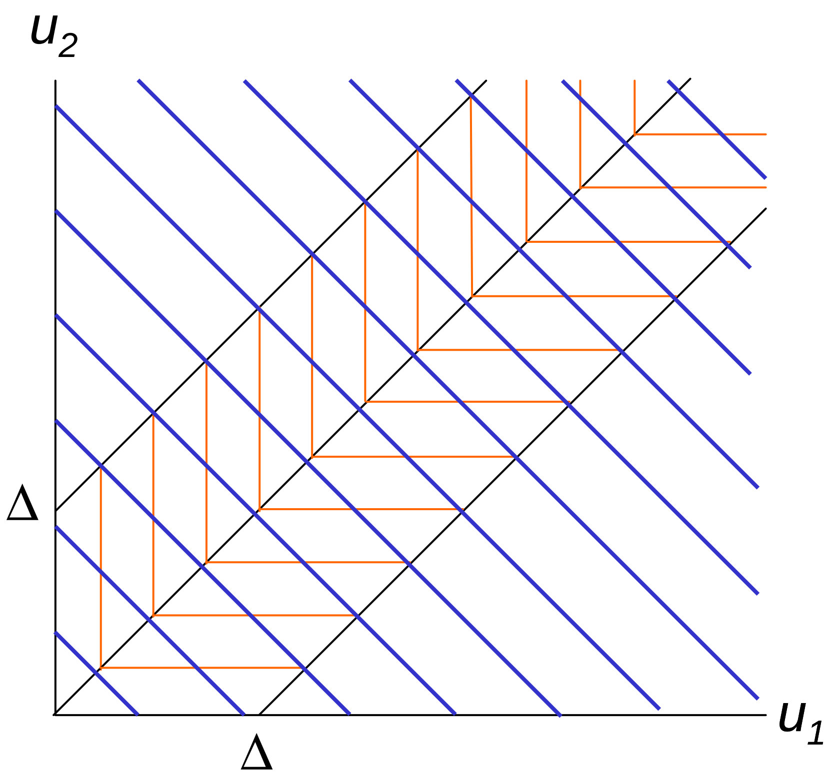


Epigraph is union of two polyhedra:

$P_1$ has recession cone

$\{(\alpha, \beta, z) \mid z \leq \alpha + \beta,\ \ \alpha, \beta \geq 0\}$

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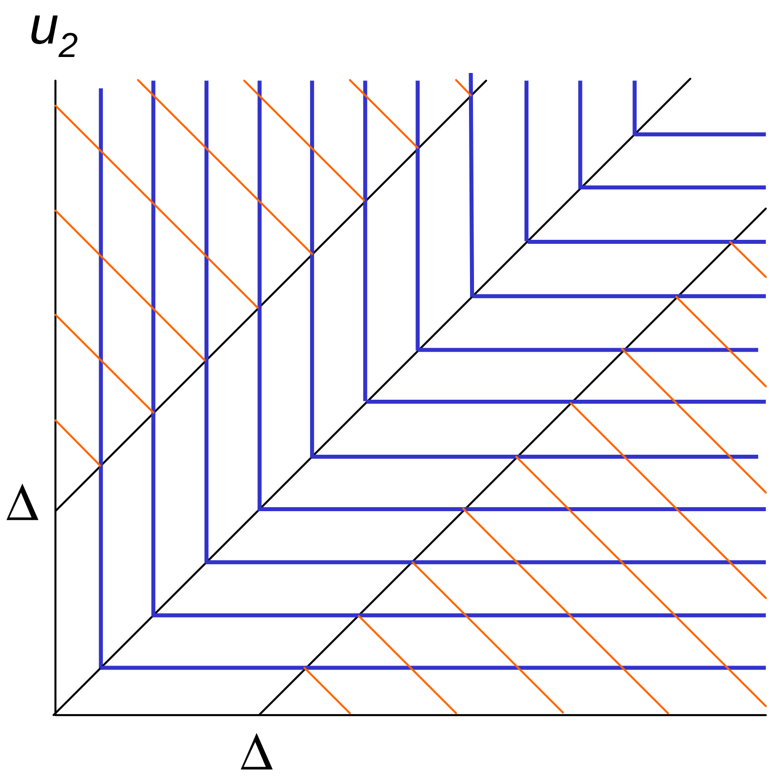


Epigraph is union of two polyhedra:

$P_1$ has recession cone

$$\{(\alpha,\beta,z)\,|\,z \le \alpha+\beta,\ \ \alpha,\beta \ge 0\}$$

$P_2$ has recession cone

$$\{(1,1,z)\,|\,0 \le z \le 2\} \cup \{(1,0,0),(0,1,0)\}$$

# Example: Health Care Benefits

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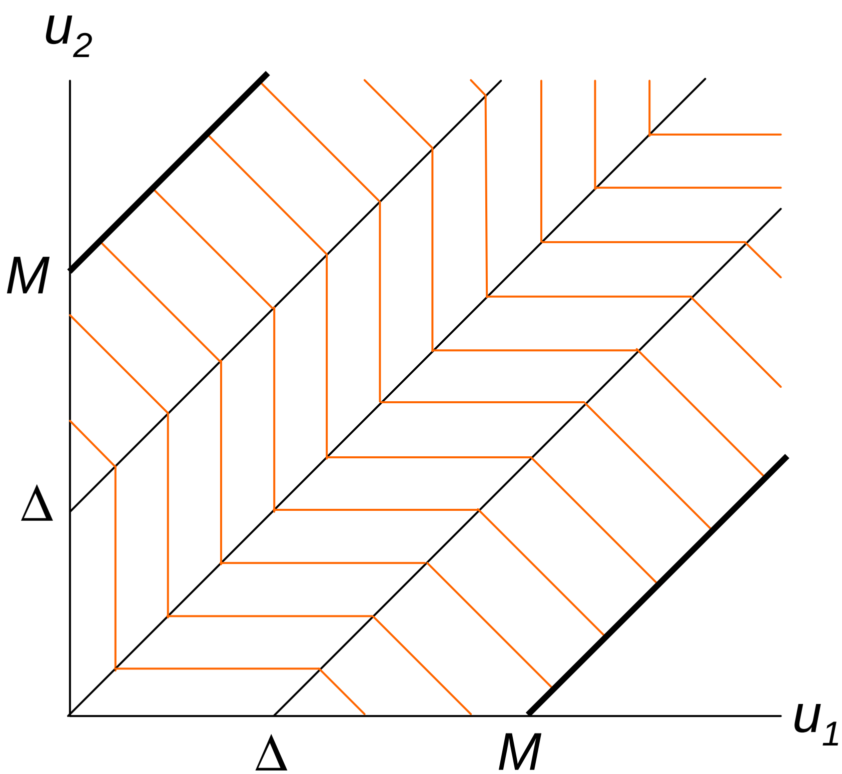
Person $i$ receives utility $u_i$.




Solution:

Add constraint $|u_1 - u_2| \leq M$

No need to bound $u_1$, $u_2$ individually

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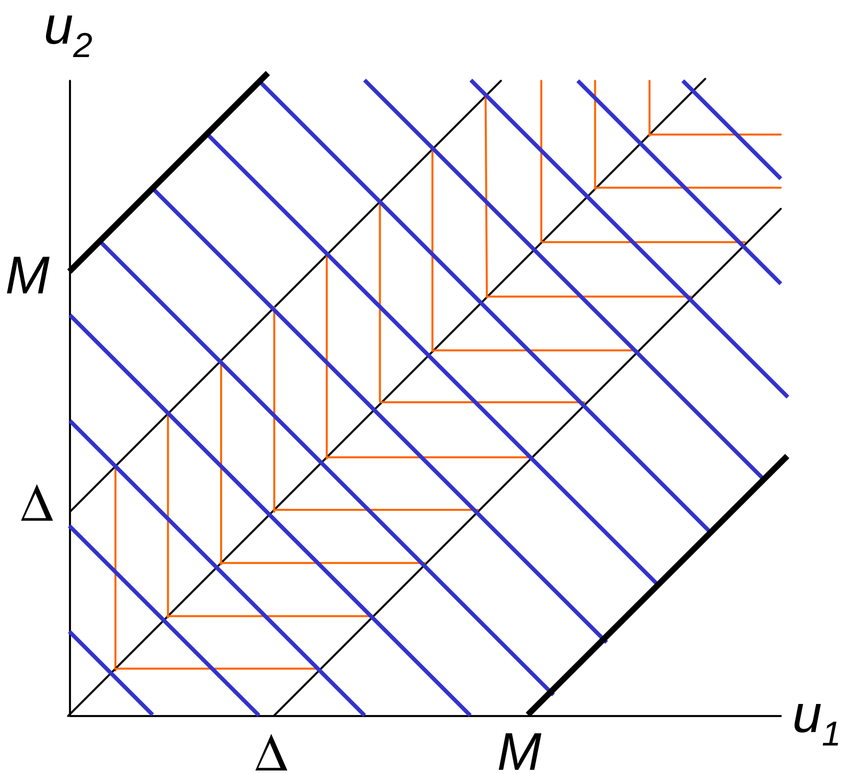


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$$\{(1,1,z) \mid 0 \le z \le 2\}$$

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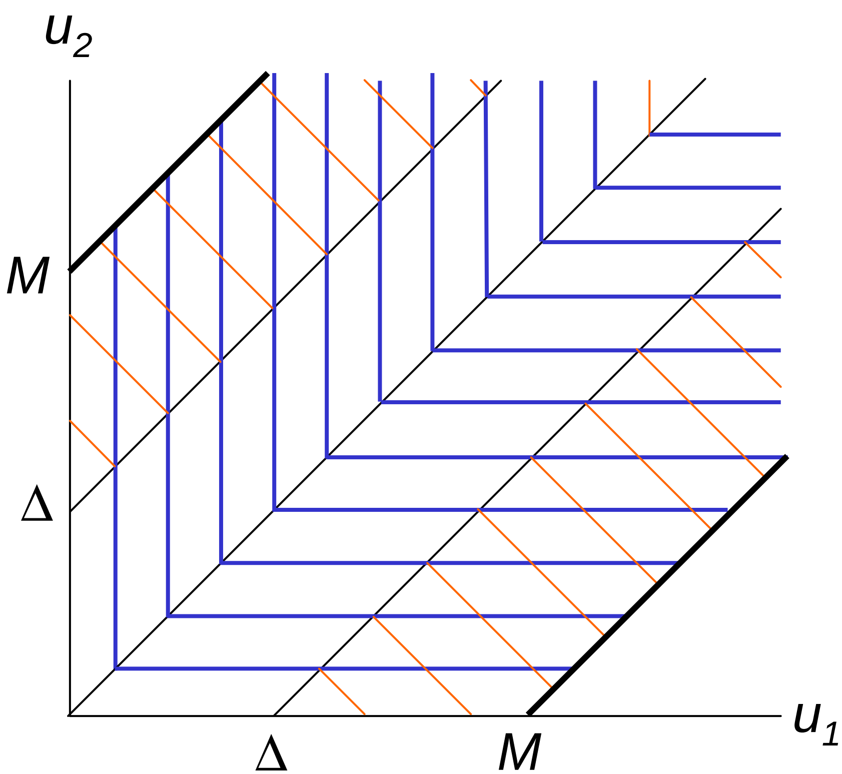


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So does $P_2$

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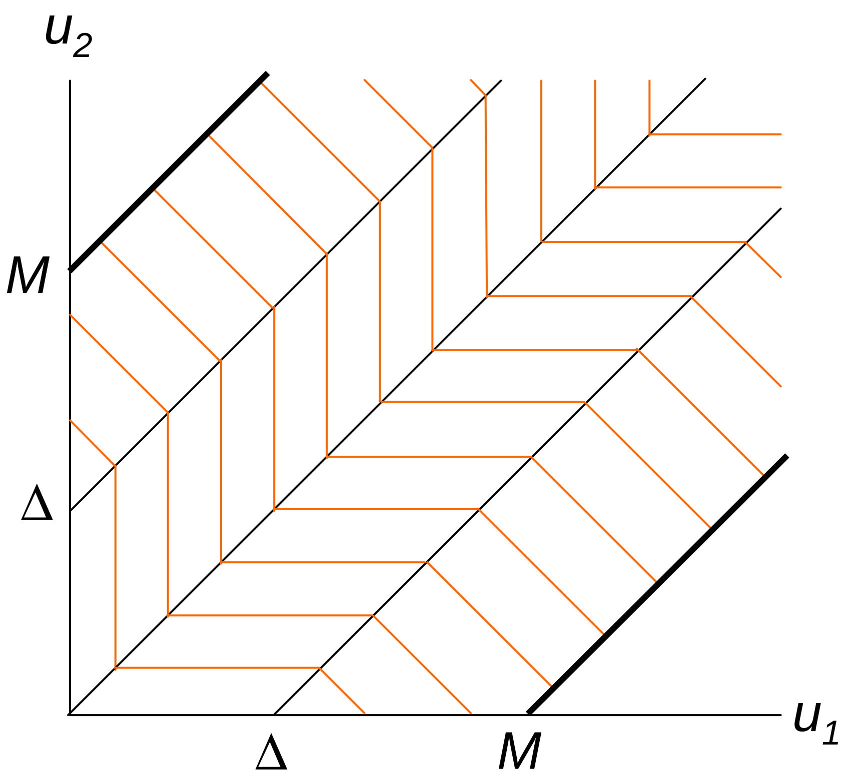


Big-$M$ model:

$$z \le 2u_1 + \Delta + (M - \Delta)y$$
$$z \le 2u_2 + \Delta + (M - \Delta)y$$
$$z \le u_1 + u_2 + \Delta(1 - y)$$
$$u_1 - u_2 \le M, \quad u_2 - u_1 \le M$$
$$u_1, u_2 \ge 0, \quad y \in \{0,1\}$$

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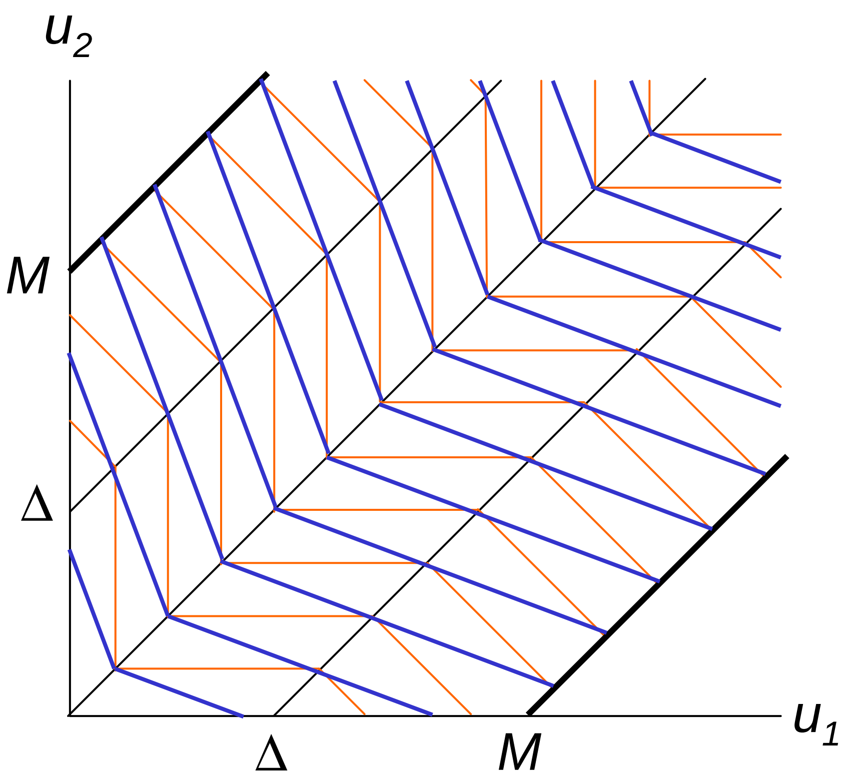


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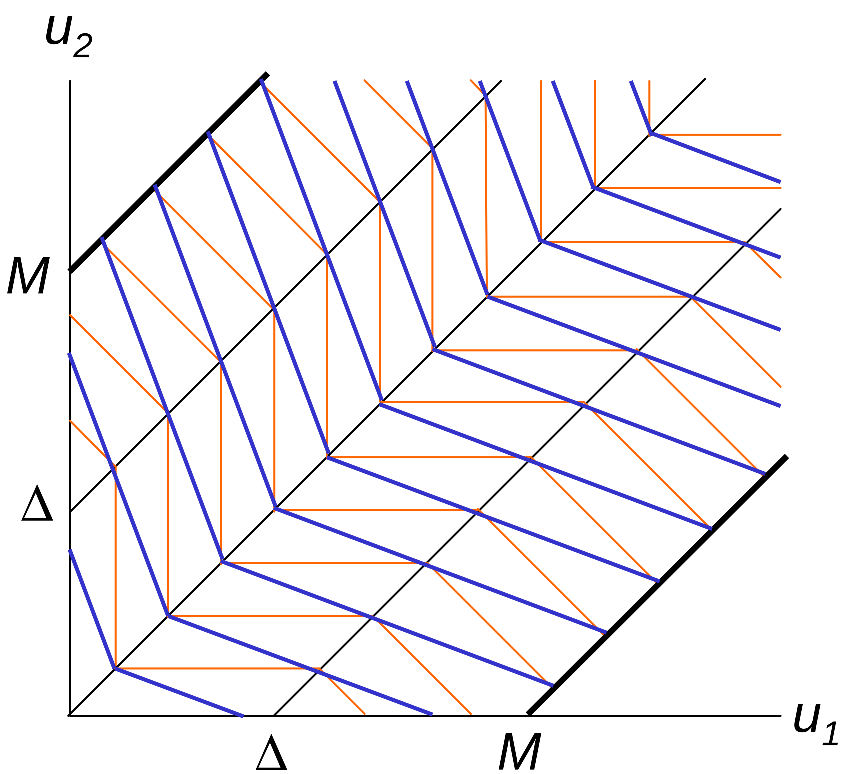
$$z \le u_1 + u_2 + \Delta(1 - y)$$
$$u_1 - u_2 \le M, \quad u_2 - u_1 \le M$$
$$u_1, u_2 \ge 0, \quad y \in \{0,1\}$$

**Theorem:** This is a convex hull formulation.

# Example: Health Care Benefits

Distribute limited health benefits to two persons.
Person *i* receives utility $u_i$.




Big-*M* model:

$$z \le 2u_1 + \Delta + (M - \Delta)y$$
$$z \le 2u_2 + \Delta + (M - \Delta)y$$
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$$u_1, u_2 \ge 0, \quad y \in \{0,1\}$$

**Theorem:** This is a convex hull formulation.

Model is no tighter if we use $u_1, u_2 \le M$

# Example: Health Care Benefits

Optimization problem for the *n*-person case:

$$\max z$$

$$z \le (n-1)\Delta + n\min_j\{u_j\} + \sum_{j=1}^{n}\max\left\{0, u_j - \min_j\{u_j\} - \Delta\right\}$$

$$|u_i - u_j| \le M, \text{ all } i, j$$

$$u \ge 0, \quad u \in S$$

# Example: Health Care Benefits

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$$u \ge 0, \quad u \in S$$

**Big-*M* disjunctive model:**

$$\max z$$

$$z \le -\Delta + \sum_{j=1}^{n} w_{ij}, \text{ all } i$$

$$w_{ij} \le \Delta + u_i + y_{ij}(M - \Delta), \text{ all } i, j$$

$$w_{ij} \le u_j + (1 - y_{ij})\Delta, \text{ all } i, j$$

$$y_{ii} = 0, \text{ all } i$$

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**Theorem:**
This is a convex hull formulation.

# General MILP Representability

# Knapsack Models

- Integer variables can also be used to express counting ideas.
  - This is totally different from the use of 0-1 variables to express unions of polyhedra.
- Examples:
  - Knapsack inequalities
  - Packing and covering
  - Logical clauses
  - Cost bounds

# Knapsack Models

- Disjunctive representability does not accommodate knapsack constraints in a natural way.
- Knapsack constraints are bounded MILP representable only if integer variables are bounded.
- …and only in a technical sense.
  - By regarding each integer lattice point as a polyhedron.

# General representability theorem

Integer variables can now be **unbounded**:

A subset $S$ of $\mathbb{R}^n \times \mathbb{Z}^p$ is **MILP representable** if $S$ is the projection onto $x$ of the feasible set of some MILP constraint set of the form

$$Ax + Bz + Dy \geq b$$

$$\boxed{x \in \mathbb{R}^n \times \mathbb{Z}^p}, \quad \boxed{z \in \mathbb{R}^m}$$

$$y \in \{0,1\}^q$$

Some modeling variables are continuous, some integer

Auxiliary continuous variables can be used

# General representability theorem

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Assume that $A$, $B$, $D$, $b$ consist of **rational data**

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Assume that $A$, $B$, $D$, $b$ consist of **rational data**

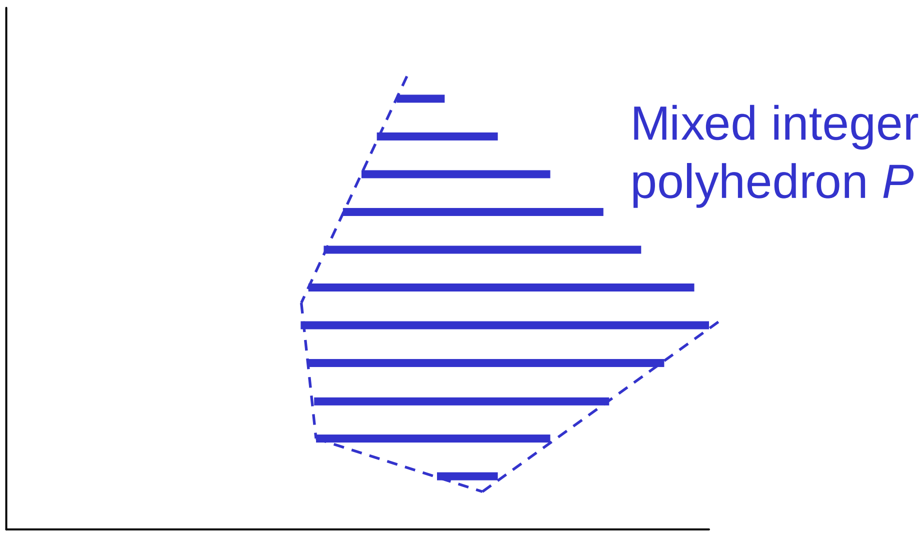
A **mixed integer polyhedron** is any set of the form

$$\left\{ x \in \mathbb{R}^n \times \mathbb{Z}^p \mid Ax \geq b \right\}$$

# General representability theorem

Rational vector *d* is a **recession direction** of a mixed integer polyhedron $P \subset \mathbb{R}^n \times \mathbb{Z}^p$ if it is a recession direction of some polyhedron $Q \subset \mathbb{R}^{n+p}$ for which

$$P = Q \cap \left(\mathbb{R}^n \times \mathbb{Z}^p\right)$$



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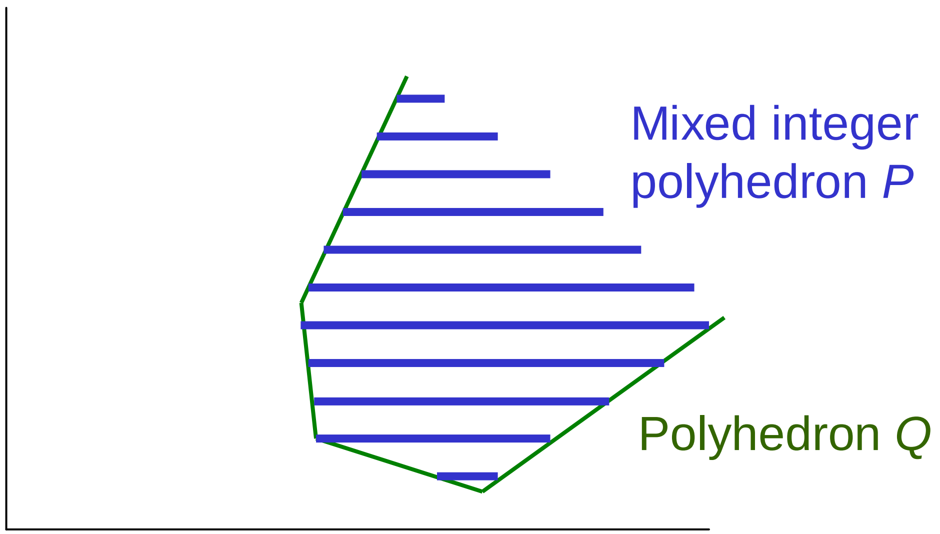

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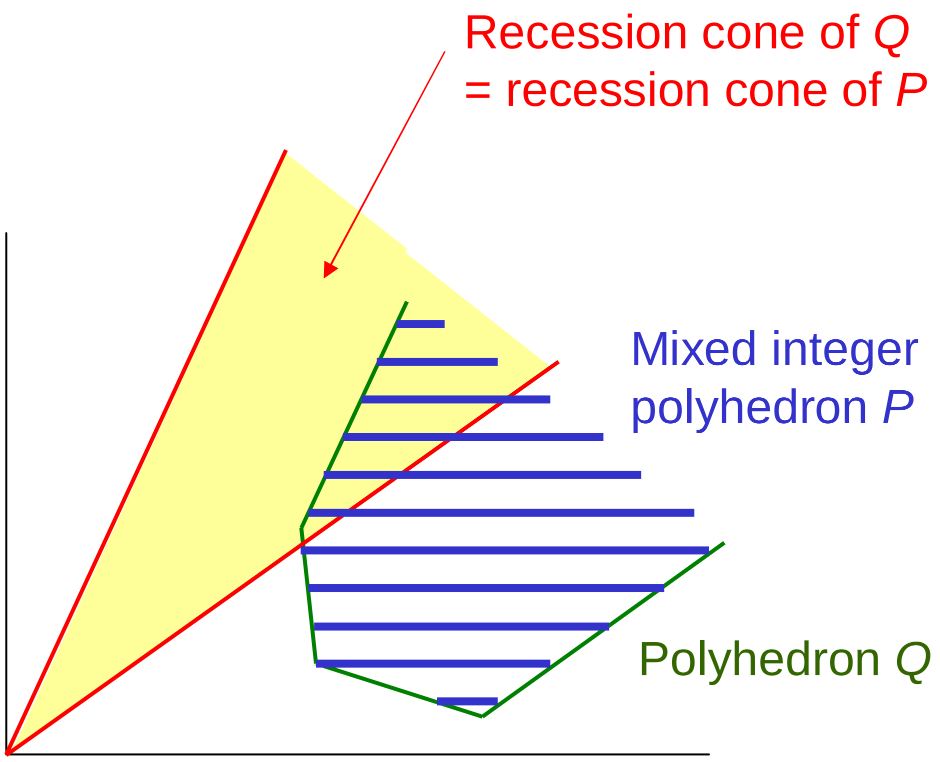
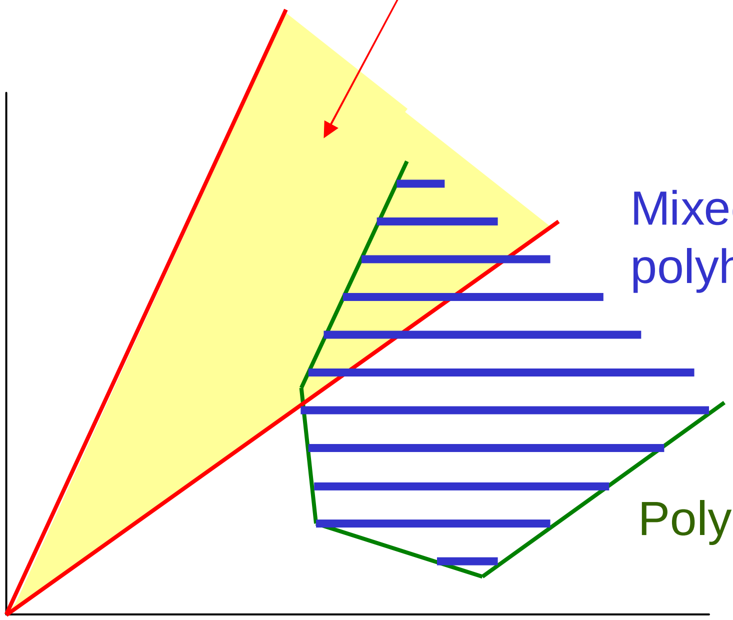


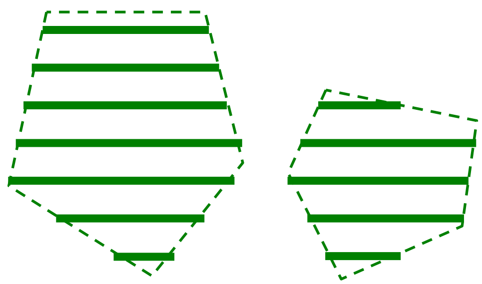
# General representability theorem

**Lemma**. All polyhedra in $\mathbb{R}^{n+p}$ having the same **nonempty** intersection with $\mathbb{R}^n \times \mathbb{Z}^p$ have the same recession cone.

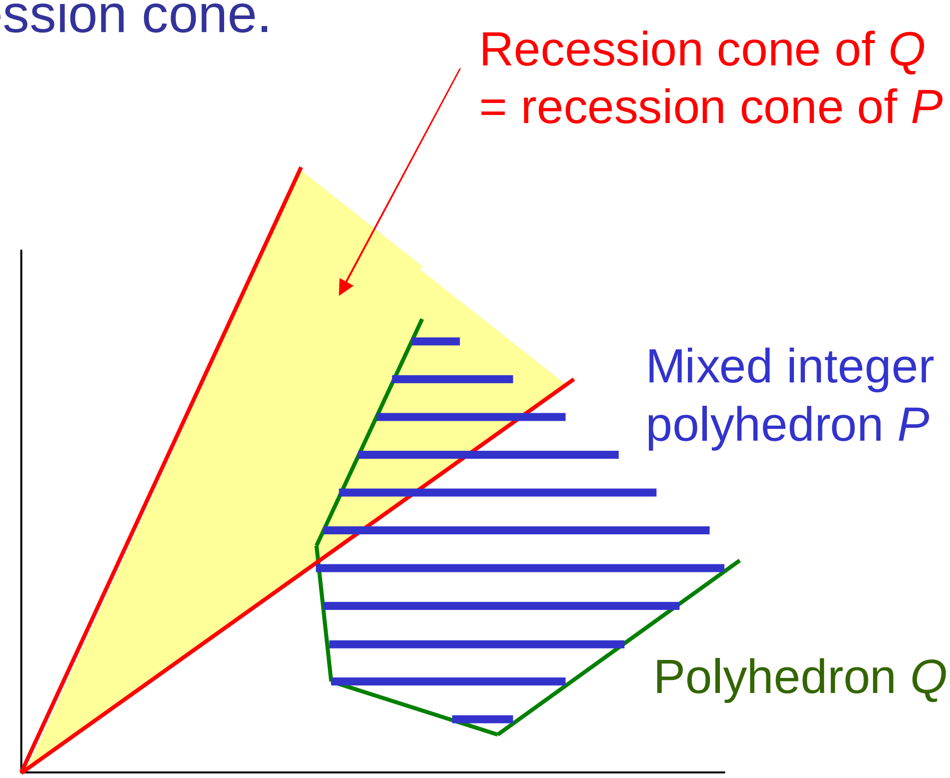


## General representability theorem

**Theorem**. A nonempty subset of $\mathbb{R}^n \times \mathbb{Z}^p$ is MILP representable if and only if it is the union of finitely many mixed integer polyhedra in $\mathbb{R}^n \times \mathbb{Z}^p$ having the same recession cone.



Union of mixed integer polyhedra with the same recession cone (in this case, the origin)



# Convex Hull Formulation

Start with a disjunction of linear systems to represent the union of mixed integer polyhedra.

$$\bigvee_k \left( A^k x \ge b^k \right)$$

The $k$th polyhedron is $\left\{ x \in \mathbb{R}^n \times \mathbb{Z}^p \mid A^k x \ge b^k \right\}$

Aside from domain of $x$, the disjunctive model is the same as before.

$$A^k x^k \ge b^k y_k, \text{ all } k$$
$$\sum_k y_k = 1$$
$$x = \sum_k x^k$$
$$x \in \mathbb{R}^n \times \mathbb{Z}^p, \; y_k \in \{0,1\}$$

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Every MILP representable set has a model of this form.

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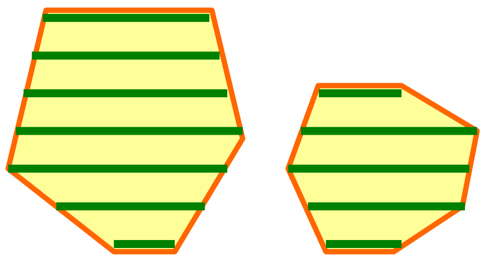
Every MILP representable set has a model of this form.

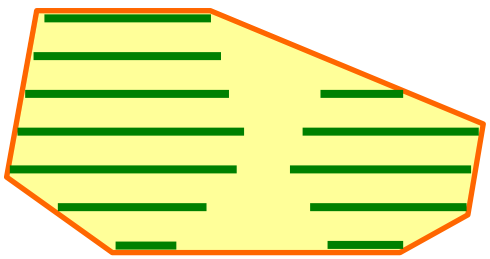
…also a model in disjunctive big-M form.

# Convex Hull Formulation

**Theorem**. If each mixed integer polyhedron has a convex hull formulation $A^k x \geq b^k$, the disjunctive model is a **convex hull** formulation of the disjunction.

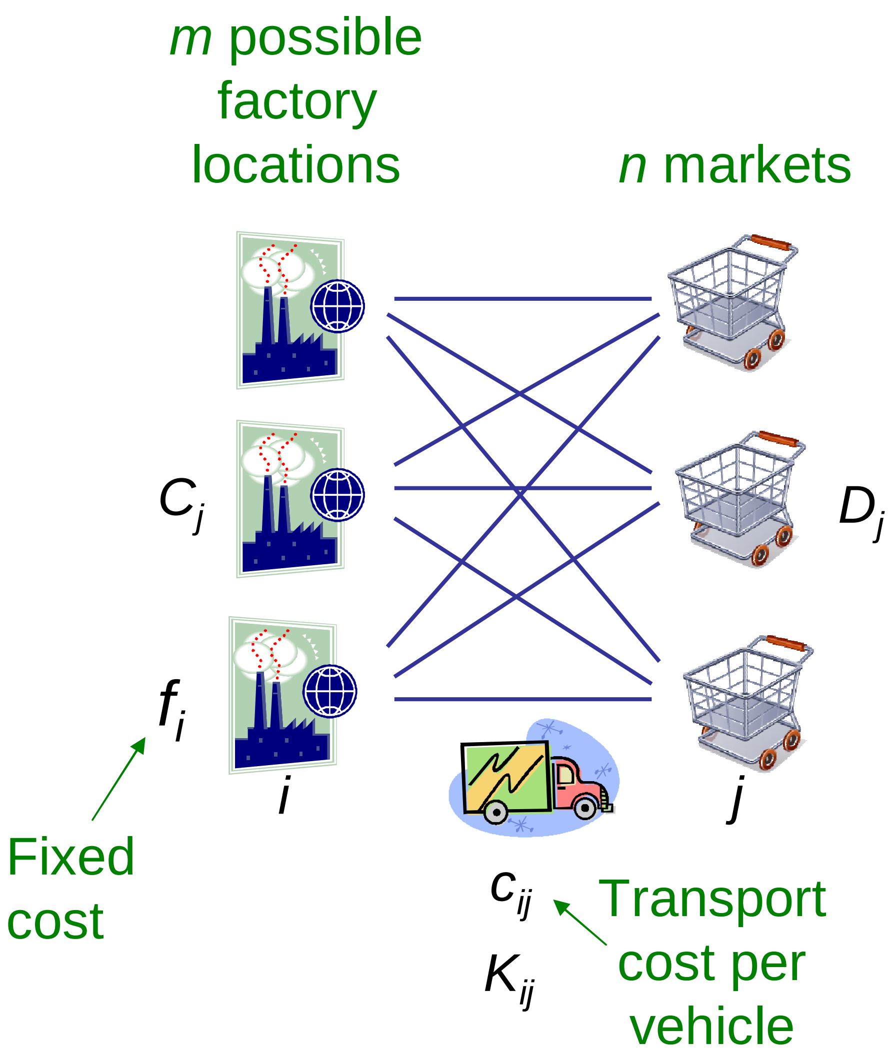


Union of mixed integer polyhedra with convex hull descriptions



Convex hull relaxation

# Example: Facility location



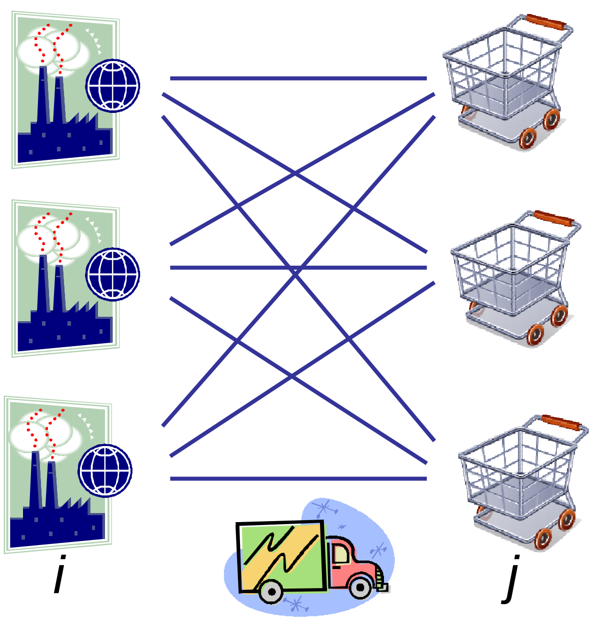

Locate factories to serve markets so as to minimize total factory cost and transport cost.

Fixed cost incurred for each vehicle used.

# Facility location

*m* possible factory locations

*n* markets



$C_j$

$f_i$

$i$ $j$

Fixed cost

$c_{ij}$

$K_{ij}$

Transport cost per vehicle

Disjunctive model:

$$\min \sum_i z_i + \sum_{ij} c_{ij} w_{ij}$$

Number of vehicles from factory *i* to market *j*

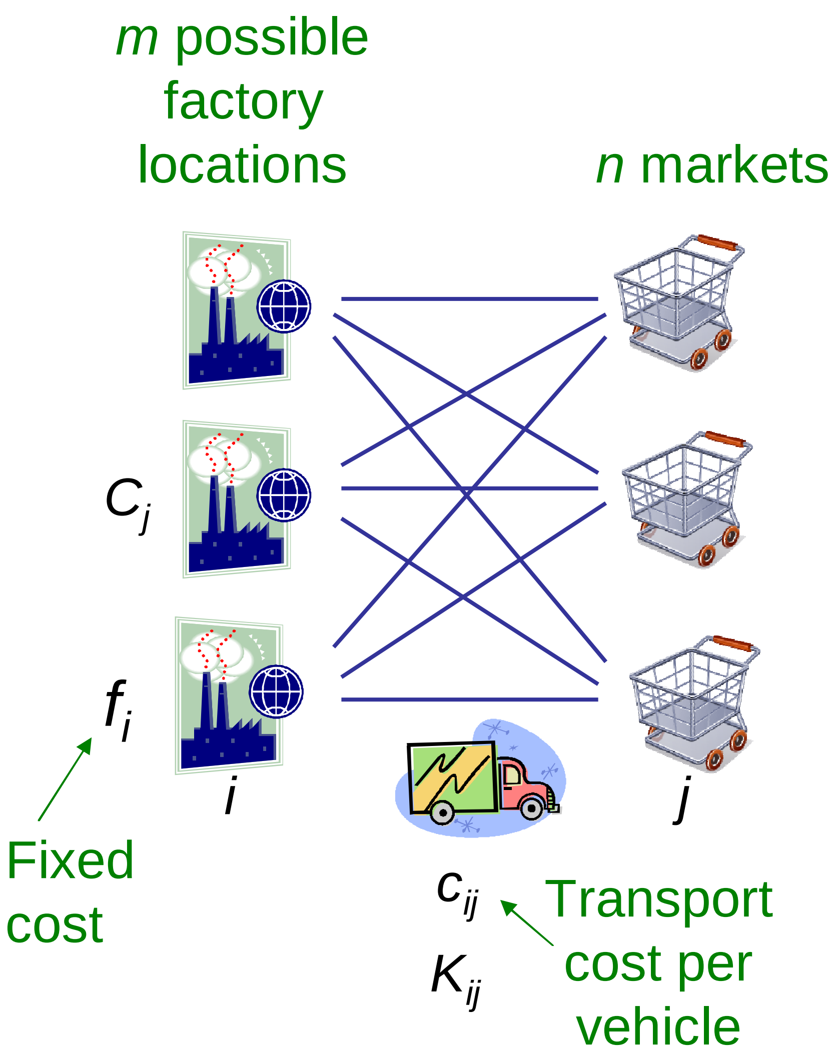
$$\begin{pmatrix} \sum_j x_{ij} \le C_i \\ 0 \le x_{ij} \le K_{ij} w_{ij}, \text{ all } j \\ z_i \ge f \\ w_{ij} \in \mathbb{Z}, \text{ all } j \end{pmatrix} \vee \begin{pmatrix} x_{ij} = 0, \text{ all } j \\ z_i = 0 \end{pmatrix}, \quad \text{all } i$$

$$\sum_i x_{ij} = D_j, \quad \text{all } j$$

Factory at location *i*

No factory at location *i*

# Facility location




Disjunctive model:

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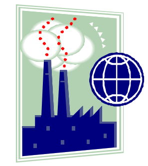
Number of vehicles from factory *i* to market *j*

Describes mixed integer polyhedron

Factory at location *i*

No factory at location *i*

# Facility location



**Disjunctive model:**

$$\min \sum_i z_i + \sum_{ij} c_{ij} w_{ij}$$

$$\begin{pmatrix} \sum_j x_{ij} \le C_i \\ 0 \le x_{ij} \le K_{ij} w_{ij}, \ \text{all } j \\ z_i \ge f \\ w_{ij} \in \mathbb{Z}, \ \text{all } j \end{pmatrix} \vee \begin{pmatrix} x_{ij} = 0, \ \text{all } j \\ z_i = 0 \end{pmatrix}, \quad \text{all } i$$

$$\sum_i x_{ij} = D_j, \quad \text{all } j$$

**MILP formulation:**

$$\min \sum_i f_i y_i + \sum_{ij} c_{ij} w_{ij}$$

$$\sum_j x_{ij} \le C_i y_i, \ \text{all } i$$

$$\sum_i x_{ij} = D_j, \ \text{all } j$$

$$0 \le x_{ij} \le K_{ij} w_{ij}, \ \text{all } i, j$$

$$y_i \in \{0,1\}, \quad w_{ij} \in \mathbb{Z}, \ \text{all } i, j$$

# Why a Single Recession Cone

Suppose $S$ is represented by

$$Ax + Bz + Dy \ge b$$
$$x \in \mathbb{R}^n \times \mathbb{Z}^p, \quad z \in \mathbb{R}^m$$
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For each binary $\bar{y}$, this describes a mixed integer polyhedron $P(\bar{y})$.

So $S$ is a union of mixed integer polyhedra.

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$$\left\{ \begin{bmatrix} x \\ u \\ y \end{bmatrix} \in \mathbb{R}^n \times \mathbb{Z}^p \times \mathbb{R}^{m+q} : \begin{bmatrix} A & B & D \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ u \\ y \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \bar{y} \end{bmatrix} \right\}$$

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That is, iff

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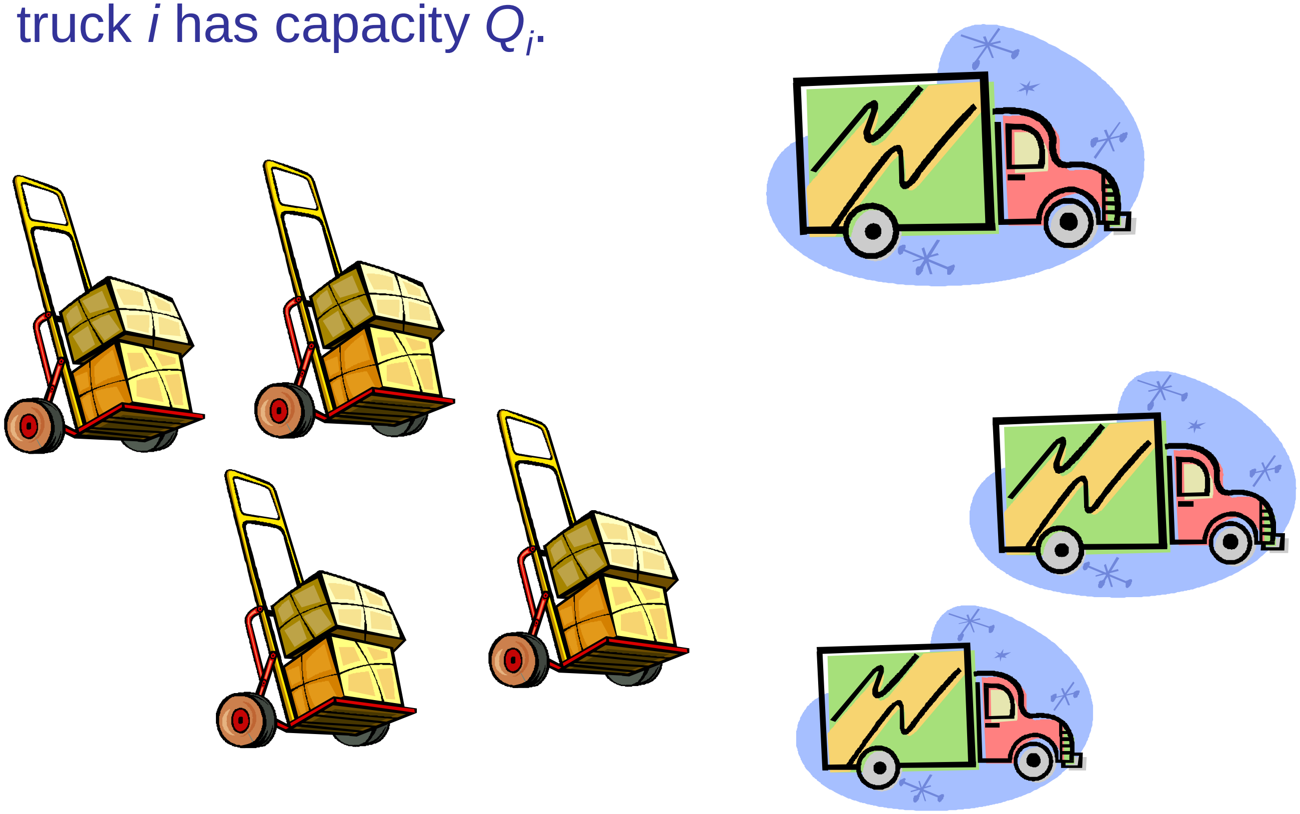
But this is independent of $y$.

# Example: Freight Packing and Transfer

- Transport packages using $n$ trucks
- Each package $j$ has size $a_j$.
- Each truck $i$ has capacity $Q_i$.



# Knapsack component

The trucks selected must have enough capacity to carry the load.

$$\sum_{i=1}^{n} Q_i \boxed{y_i} \geq \sum_j a_j$$

= 1 if truck *i* is selected

# Disjunctive component

Truck $i$ selected | Truck $i$ not selected

Cost variable → $z_i$

Cost of operating truck $i$ → $c_i$

$$\begin{pmatrix} z_i \geq c_i \\ \sum_j a_j x_{ij} \leq Q_i \\ x_{ij} \in \{0,1\},\ \text{all } j \end{pmatrix} \vee \begin{pmatrix} z_i \geq 0 \\ x_{ij} = 0 \end{pmatrix}$$

$x_{ij}$ = 1 if package $j$ is loaded on truck $i$

# Disjunctive component

Truck $i$ selected

Truck $i$ not selected

$$\begin{pmatrix} z_i \ge c_i \\ \sum_j a_j x_{ij} \le Q_i \\ x_{ij} \in \{0,1\}, \text{ all } j \end{pmatrix} \vee \begin{pmatrix} z_i \ge 0 \\ x_{ij} = 0 \end{pmatrix}$$

Cost variable

Cost of operating truck $i$

Describes mixed integer polyhedron

= 1 if package $j$ is loaded on truck $i$

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Convex hull MILP formulation

$$z_i \ge c_i y_i$$

$$\sum_j a_j x_{ij} \le Q_i y_i$$

$$0 \le x_{ij} \le y_i$$

## The resulting model

$$\min \sum_{i=1}^{n} c_i y_i$$

$$\sum_j a_j x_{ij} \le Q_i y_i, \quad \text{all } i$$

$$0 \le x_{ij} \le y_i, \quad \text{all } i, j$$

Disjunctive component

$$\sum_{i=1}^{n} x_{ij} = 1, \quad \text{all } j$$

Logical condition
(each package must be shipped)

$$\sum_{i=1}^{n} Q_i y_i \ge \sum_j a_j$$

Knapsack component

$$x_{ij}, y_i \in \{0,1\}$$

## The resulting model

$$\min \sum_{i=1}^{n} c_i y_i$$

$$\sum_j a_j x_{ij} \le Q_i y_i, \quad \text{all } i$$

$$0 \le x_{ij} \le y_i, \quad \text{all } i, j$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad \text{all } j$$

$$\sum_{i=1}^{n} Q_i y_i \ge \sum_j a_j$$

$$x_{ij}, y_i \in \{0,1\}$$

The $y_i$ is redundant but makes the continuous relaxation tighter.

This is a modeling “trick,” part of the folklore of modeling.

## The resulting model

$$\min \sum_{i=1}^{n} c_i y_i$$

$$\sum_j a_j x_{ij} \leq Q_i \boxed{y_i}, \quad \text{all } i$$

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$$\sum_{i=1}^{n} x_{ij} = 1, \quad \text{all } j$$

$$\boxed{\sum_{i=1}^{n} Q_i y_i \geq \sum_j a_j}$$

$$x_{ij}, y_i \in \{0,1\}$$

The $y_i$ is redundant but makes the continuous relaxation tighter.

This is a modeling "trick," part of the folklore of modeling.

Conventional modeling wisdom would not use this constraint, because it is the sum of the first constraint over i.

But it radically reduces solution time, because it generates lifted knapsack cuts.

# Research issues

- Can the **simplification** of a convex hull MILP formulation be automated?
- What are some conditions under which a **big-M** disjunctive model is a convex hull formulation?
- When does convex hull formulation of **logical constraints** tighten the model?
- How can a **modeling system** facilitate and encourage principled modeling?