Decomposition Methods for Integer Linear Programming

Matthew Galati[1] Ted Ralphs[2]

[1]SAS Institute, Advanced Analytics, Operations Research R & D

[2]COR@L Lab, Department of Industrial and Systems Engineering, Lehigh University

Ph.D. Defense
Industrial and Systems Engineering
Lehigh University, Bethlehem, PA

The Decomposition Principle in Integer Programming

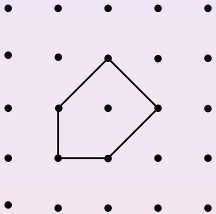
Basic Idea: By leveraging our ability to solve the optimization/separation problem for a relaxation, we can improve the bound yielded by the LP relaxation.

$$z_{\mathrm{IP}} = \min_{x \in \mathbb{Z}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\}$$

$$z_{\mathrm{LP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid A'x \geq b', A''x \geq b'' \right\}$$

$$z_{\mathrm{D}} = \min_{x \in \mathcal{P}'} \left\{ c^\top x \mid A''x \geq b'' \right\}$$

$$z_{\mathrm{IP}} \geq z_{\mathrm{D}} \geq z_{\mathrm{LP}}$$



$\mathcal{P} = \mathrm{conv}\{x \in \mathbb{Z}^n \mid A'x \geq b', A''x \geq b''\}$

Assumptions:

- $\mathrm{OPT}(\mathcal{P}, c)$ and $\mathrm{SEP}(\mathcal{P}, x)$ are *"hard"*
- $\mathrm{OPT}(\mathcal{P}', c)$ and $\mathrm{SEP}(\mathcal{P}', x)$ are *"easy"*
- $\mathcal{Q}''$ can be represented explicitly (description has polynomial size)
- $\mathcal{P}'$ must be represented implicitly (description has exponential size)

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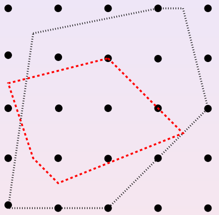

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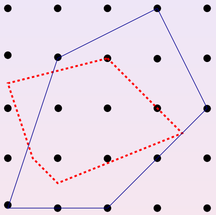

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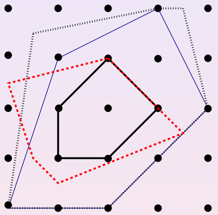

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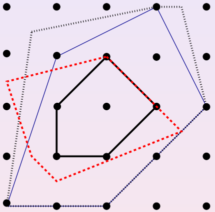

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Example - Traveling Salesman Problem (TSP)

Traveling Salesman Problem Formulation

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\begin{array}{lll}
x(\delta(\{u\})) & = & 2 \qquad \forall u \in V \\
x(E(S)) & \leq & |S| - 1 \qquad \forall S \subset V,\ 3 \leq |S| \leq |V| - 1 \\
x_e \in \{0,1\} & & \qquad \forall e \in E
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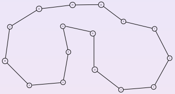

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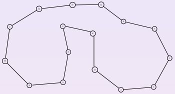
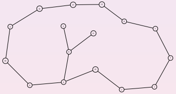


Two possible decompositions

Find a spanning subgraph with $|V|$ edges that satisfies the 2-degree constraints ($\mathcal{P}' =$ 1-Tree)

$$\begin{array}{lcll} x(\delta(\{0\})) & = & 2 & \\ x(E(V)) & = & |V| & \\ x(E(S)) & \leq & |S|-1 & \forall S \subset V \setminus \{0\}, 3 \leq |S| \leq |V|-1 \\ x_e \in \{0,1\} & & & \forall e \in E \end{array}$$



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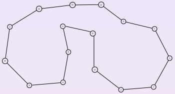

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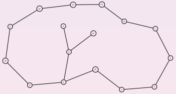

Find a 2-matching that satisfies the subtour constraints ($\mathcal{P}' =$ 2-Matching)

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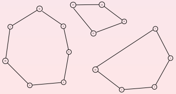

Outline

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Thesis Contributions

- *Conceptual framework tying together numerous decomposition-based methods for generating approximations of the convex hull of feasible solutions.*
  - Traditional method for outer approximation: **cutting plane method**
  - Traditional methods for inner approximations: **Dantzig-Wolfe method** and **Lagrangian method**
  - Integrated methods: **price-and-cut** and **relax-and-cut**
- *Introduction to a relatively new integrated method called* **decompose-and-cut**, *an associated class of cutting planes called* **decomposition cuts**, *and the concept of* **structured separation.**
- *Descriptions of numerous implementation considerations for branch-and-price-and-cut, including an introduction to a relatively unknown idea of using* **nested polyhedra** *for generating inner approximations.*
- **DIP** *(***D***ecomposition for* **I***nteger* **P***rogramming), an extensible open-source software framework for implementing decomposition-based methods with minimal user burden.*
- *MILPBlock, a DIP application and generic black-box solver for* **block-diagonal** *MILPs that fully automates the branch-and-price-and-cut algorithm with no additional user input.*
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CPM builds an *outer* approximation of $\mathcal{P}'$ intersected with $\mathcal{Q}''$

- **Master**: $z_{\mathrm{CP}} = \min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid Dx \geq d, A''x \geq b'' \right\}$
- **Subproblem**: $\mathrm{SEP}(\mathcal{P}', x_{\mathrm{CP}})$

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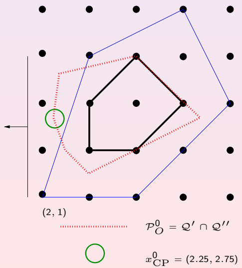
exponential number of constraints



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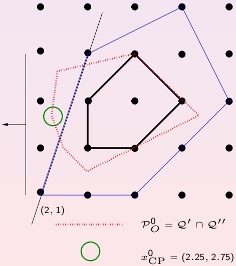

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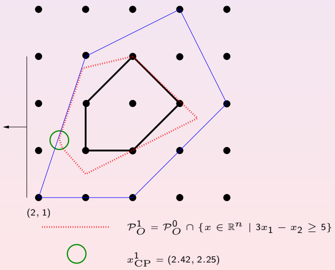

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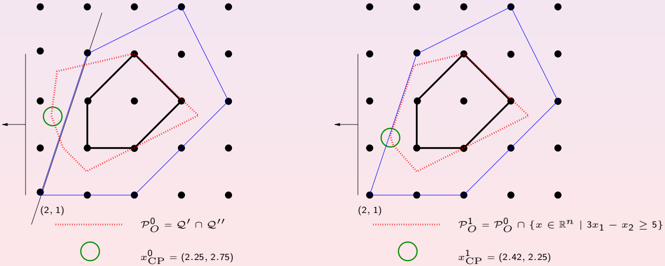

Dantzig-Wolfe Method (DW)

DW builds an *inner* approximation of $\mathcal{P}'$ intersected with $\mathcal{Q}''$

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- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{DW}}^\top A''\right)$

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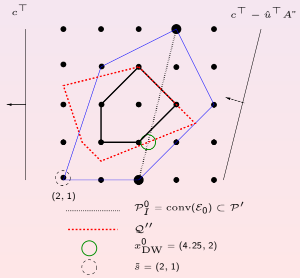

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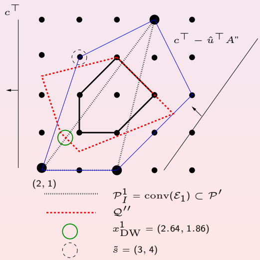

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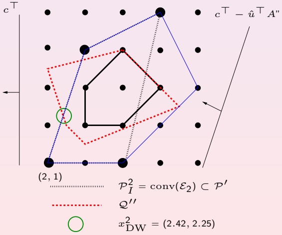

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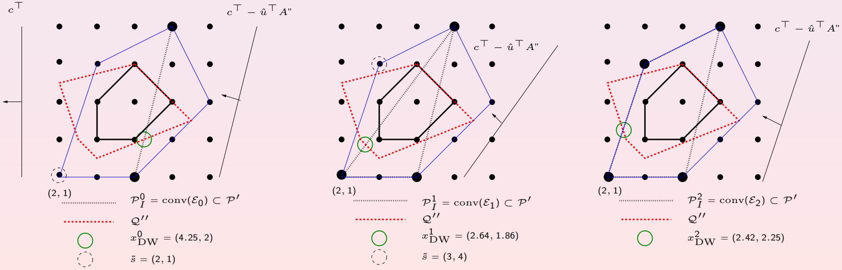
$$\mathcal{P}' = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{s \in \mathcal{E}} s\lambda_s, \sum_{s \in \mathcal{E}} \lambda_s = 1, \lambda_s \geq 0 \; \forall s \in \mathcal{E} \right\}$$

exponential number of variables



Lagrangian Method (LD)

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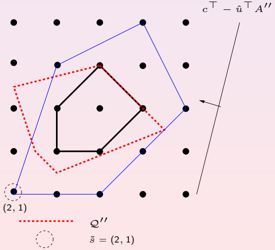

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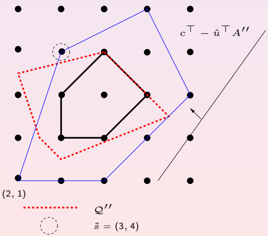

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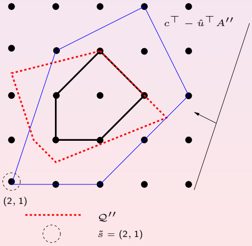

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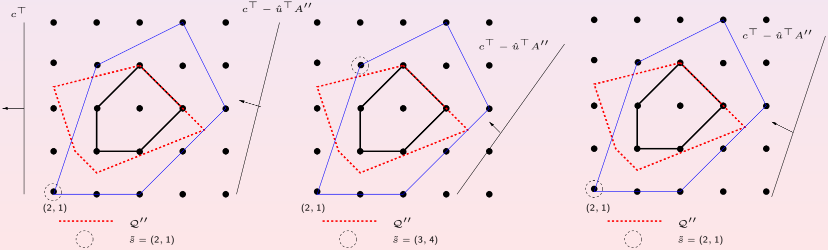
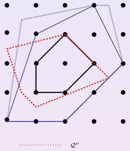


Common Threads

- The LP bound is obtained by optimizing over the intersection of two explicitly defined polyhedra.

$$z_{\mathrm{LP}} = \min_{x \in \mathbb{R}^n} \{c^\top x \mid x \in \mathcal{Q}' \cap \mathcal{Q}''\}$$



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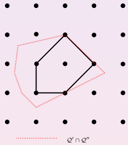

- Traditional decomp-based bounding methods contain two primary steps
  - **Master Problem:** Update the primal/dual solution information
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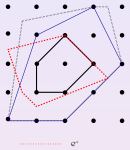

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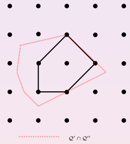

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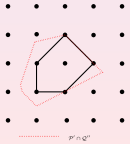

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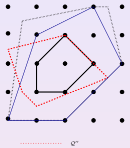

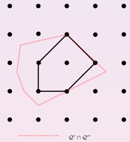

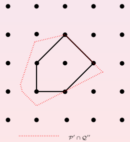

Price-and-Cut Method (PC)

PC approximates $\mathcal{P}$ by building an *inner* approximation of $\mathcal{P}'$ (as in DW) intersected with an *outer* approximation of $\mathcal{P}$ (as in CPM)

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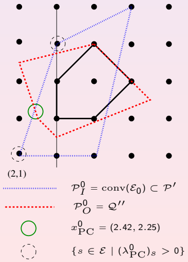
- **Subproblem**: $\mathrm{OPT}\left(\mathcal{P}', c^\top - u_{\mathrm{PC}}^\top D\right)$ or $\mathrm{SEP}\left(\mathcal{P}, x_{\mathrm{PC}}\right)$

- As in CPM, separate $\hat{x}_{\mathrm{PC}} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$ from $\mathcal{P}$ and add cuts to $[D, d]$.
- **Key Idea**: Cut generation takes place in the space of the compact formulation, maintaining the structure of the column generation subproblem.



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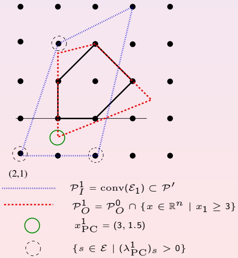

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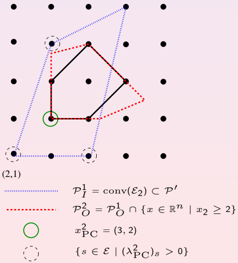

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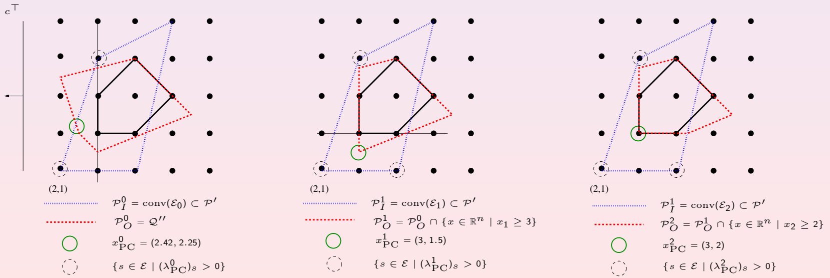

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RC approximates $\mathcal{P}$ by tracing an *inner* approximation of $\mathcal{P}'$ (as in LD) penalizing points outside of a dynamically generated *outer* approximation of $\mathcal{P}$ (as in CPM)

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- In each iteration, separate $\hat{s} \in \mathcal{E}$, a solution to the Lagrangian relaxation.
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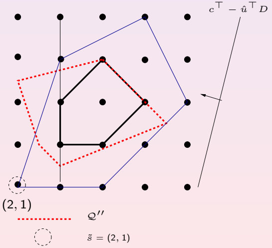

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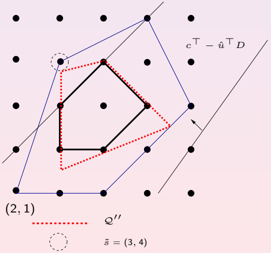

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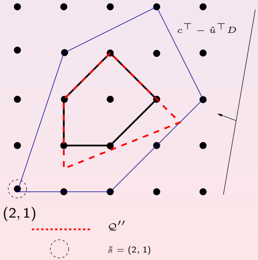

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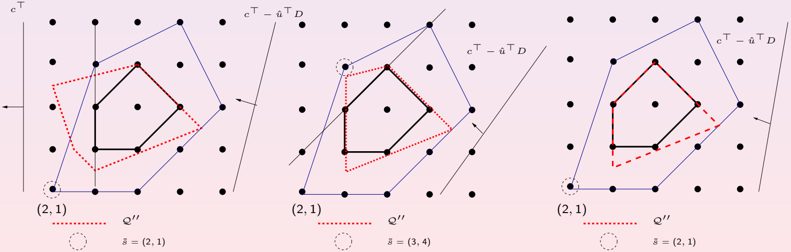

Structured Separation

- In general, the complexity of $\mathrm{OPT}(X, c) = \mathrm{SEP}(X, x)$.
- **Observation:** Restrictions on input or output can change their complexity.
- Template Paradigm, restricts the *output* of $\mathrm{SEP}(X, x)$ to valid inequalities that conform to a certain structure. This class of inequalities forms a polyhedron $\mathcal{C} \supset X$ (the *closure*).
- For example, let $\mathcal{P}$ be the convex hull of solutions to the TSP.
  - $\mathrm{SEP}(\mathcal{P}, x)$ is $\mathcal{NP}$-Complete.
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    - $\mathcal{P}^{\mathrm{Subtour}}$, the Subtour Polytope (separation using Min-Cut), or
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- Separation of Subtour Inequalities:

$$x(E(S)) \leq |S| - 1$$

- $\mathrm{SEP}(\mathcal{P}^{\mathrm{Subtour}}, x)$ for $x \in \mathbb{R}^n$ can be solved in $O\left(|E||V| + |V|^2 \log |V|\right)$ (Min-Cut)
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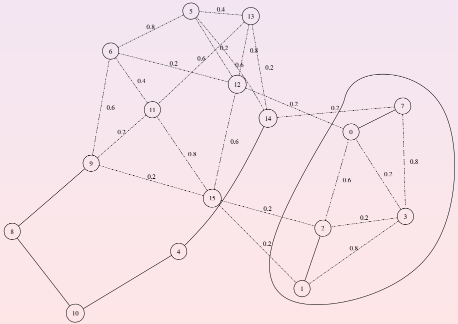

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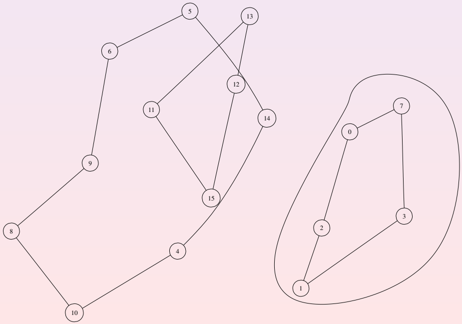

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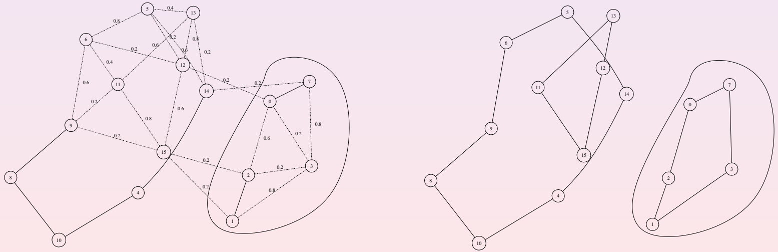

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- Separation of Comb Inequalities:

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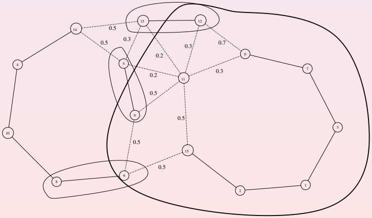

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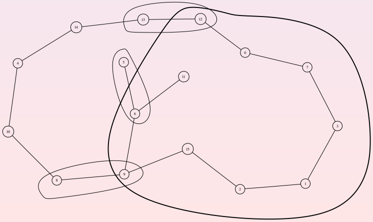

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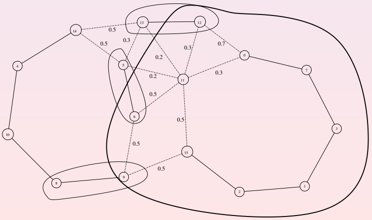

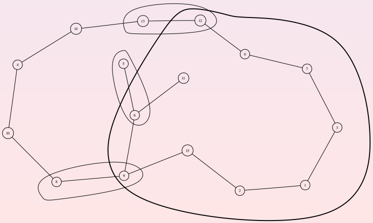

Motivation

- In Relax-and-Cut, solutions to the Lagrangian subproblem $s \in \mathcal{E}$ typically have some *nice* combinatorial structure. So, in RC, $\mathrm{SEP}(\mathcal{P}, s)$, can be relatively easy as opposed to general separation.
- **Question:** Can we take advantage of this in other contexts?
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- This condition is difficult to verify, so we typically use the *necessary condition* that the generated inequality be violated by some member of that face, $x \in F$.
  - In CPM, we solve $\mathrm{SEP}(\mathcal{P}, x^t_{\mathrm{CP}})$, where $x^t_{\mathrm{CP}} \in F^t$, and F^t is optimal face over $\mathcal{P}^t_O \cap \mathcal{Q}''$
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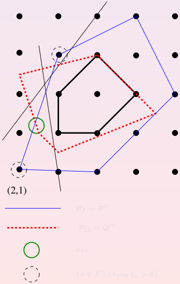

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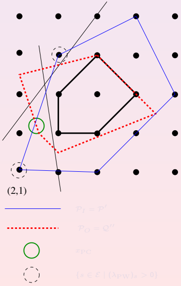

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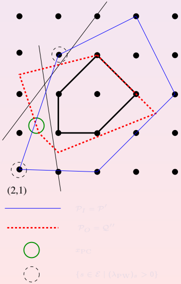

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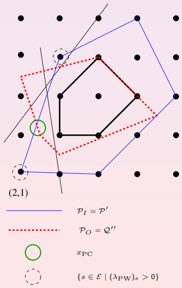

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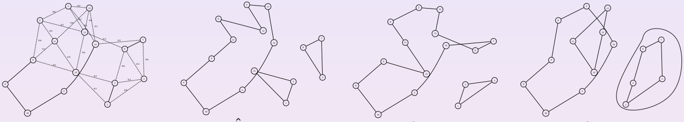

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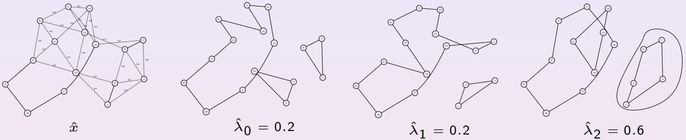

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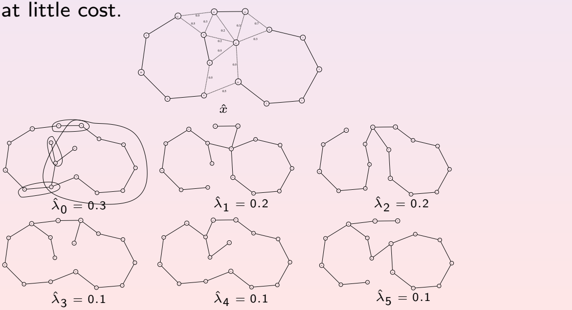

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Decompose-and-Cut: Each iteration of CPM, decompose into convex combo of e.p.'s of $\mathcal{P}'$

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- If $\hat{x}_{\mathrm{CP}}$ lies outside $\mathcal{P}'$ the decomposition will fail
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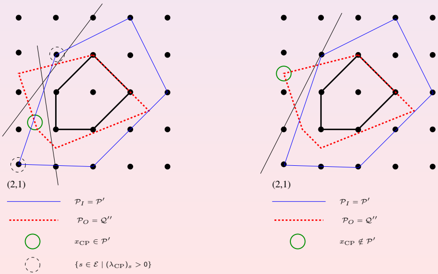
Decomposition Cuts

$$\begin{aligned} u_{\mathrm{DC}}^t s + \alpha_{\mathrm{DC}}^t &\le 0 \;\forall s \in \mathcal{P}' \quad \text{and} \\ u_{\mathrm{DC}}^t \hat{x}_{\mathrm{CP}} + \alpha_{\mathrm{DC}}^t &> 0 \end{aligned}$$



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  - Later used in TSP **Concorde** by ABCC (*non-template cuts*)
  - Now being used (in some form) for generic MILP by **Gurobi**
- This tells us that our cuts are *missing something* related to $\mathcal{P}'$
- The machinery for solving this already exists (=column generation)
- Much easier than DW problem because *feasibility* problem and
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Branching for Inner Methods (PC)

- Add column bounds to $[A'', b'']$ and map back to the compact space $\hat{x} = \sum_{s \in \mathcal{E}} s \hat{\lambda}_s$
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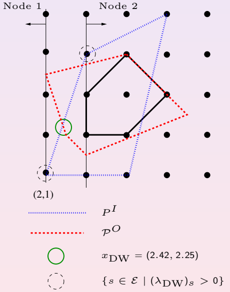

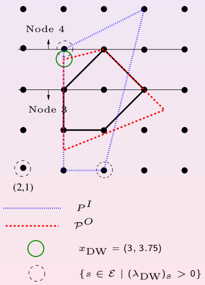

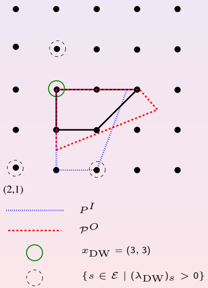

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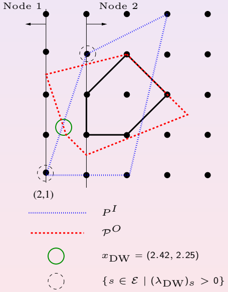

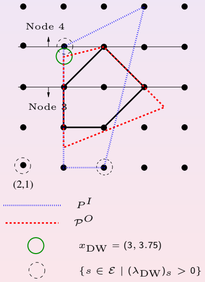

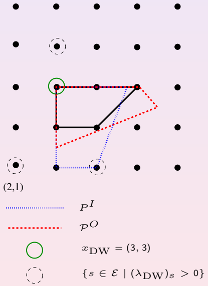

$$\begin{array}{lrl} \text{Node 1:} & 4\lambda_{(4,1)} + 5\lambda_{(5,5)} + 2\lambda_{(2,1)} + 3\lambda_{(3,4)} \leq & 2 \\ \text{Node 2:} & 4\lambda_{(4,1)} + 5\lambda_{(5,5)} + 2\lambda_{(2,1)} + 3\lambda_{(3,4)} \geq & 3 \end{array}$$

Branching for Inner Methods (RC)

- In general, Lagrangian methods do *not* provide a primal solution λ
- Let $\mathcal{B}$ define the extreme points found in solving subproblems for z_{LD}
- Build an inner approximation using this set, then proceed as in PC

$$\mathcal{P}_I = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{s \in \mathcal{B}} s\lambda_s, \sum_{s \in \mathcal{B}} \lambda_s = 1, \lambda_s \geq 0 \;\forall s \in \mathcal{B} \right\}$$

$$\min_{\lambda \in \mathbb{R}^{\mathcal{B}}_+} \left\{ c^\top \left(\sum_{s \in \mathcal{B}} s\lambda_s \right) \;\middle|\; A'' \left(\sum_{s \in \mathcal{B}} s\lambda_s \right) \geq b'', \sum_{s \in \mathcal{B}} \lambda_s = 1 \right\}$$

- Closely related to *volume* algorithm and *bundle* methods

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Relaxation Separability

- Key motivation of original DW was separability of the subproblem
- Independence lends itself nicely to **parallel implementation**
- Projections into subspace for each block, then take e.p.'s from each

$$
\begin{pmatrix}
A''_1 & A''_2 & \cdots & A''_\kappa \\
\hline
A'_1 & & & \\
 & A'_2 & & \\
 & & \ddots & \\
 & & & A'_\kappa
\end{pmatrix}
$$

$$
\begin{aligned}
\mathcal{P}'_k &= \operatorname{conv}\{x \in \mathbb{Z}^n \mid A'_k x \geq b'_k, x_i = 0 \ \forall i \in K \setminus \{k\}\} \\
\mathcal{P}' &= \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{k \in K} \sum_{s \in \mathcal{E}^k} s \lambda^k_s, \sum_{k \in K} \sum_{s \in \mathcal{E}^k} \lambda^k_s = 1 \ \forall k \in K, \lambda^k_s \geq 0 \ \forall k \in K,\ s \in \mathcal{E}^k \right\} \\
z_{\mathrm{DW}} &= \min_{\lambda} \left\{ c^\top \left(\sum_{k \in K} \sum_{s \in \mathcal{E}^k} s \lambda^k_s \right) \;\middle|\; A'' \left(\sum_{k \in K} \sum_{s \in \mathcal{E}^k} s \lambda^k_s \right) \geq b'', \sum_{s \in \mathcal{E}^k} \lambda^k_s = 1 \ \forall k \in K \right\}
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Generalized Assignment Problem (GAP)

$$
\begin{array}{llll}
\min & \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} & & \\
 & \sum_{j \in N} w_{ij} x_{ij} \leq & b_i & \forall i \in M \\
 & \sum_{i \in M} x_{ij} = & 1 & \forall j \in N \\
 & x_{ij} \in & \{0,1\} & \forall i,j \in M \times N
\end{array}
$$

Identical Subproblems

- One motivation for using inner methods like DW is to break symmetry
- Block-diagonal special-case: *identical subproblems*

Vehicle Routing Problem (VRP)

$$
\begin{array}{llll}
\min & \displaystyle\sum_{k \in M} \sum_{(i,j) \in A} c_{ij} x_{ijk} & & \\
& \displaystyle\sum_{k \in M} \sum_{j \in N} x_{ijk} & = 1 & \forall i \in V \\
& \displaystyle\sum_{i \in V} \sum_{j \in N} d_i x_{ijk} & \leq C & \forall k \in M \\
& \displaystyle\sum_{j \in N} x_{0jk} & = 1 & \forall k \in M \\
& \displaystyle\sum_{i \in N} x_{ihk} - \sum_{j \in N} x_{hjk} & = 0 & \forall h \in V, k \in M \\
& \displaystyle\sum_{i \in N} x_{i,n+1,k} & = 1 & \forall k \in M \\
& x_{ijk} \in \{0,1\} & & \forall (i,j) \in A, k \in M
\end{array}
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$$\Lambda_s = \sum_{k \in K} s\lambda_s^k \ \forall s \in \mathcal{E}$$

$$z_{\mathrm{DW}} = \min_{\Lambda} \left\{ c^\top (s\Lambda_s) \;\middle|\; A'' \left(\sum_{s \in \mathcal{E}} s\Lambda_s \right) \geq b'', \sum_{s \in \mathcal{E}} \Lambda_s^k = K \right\}$$

- Aggregation step breaks our dependence on one-to-one mapping $\hat{x} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$
- Vanderbeck, et al. has been investigating disaggregation based on lexicographic ordering

Nested Polyhedra

- Outer methods use various approximations to improve the bound (template paradigm)
- **New Idea:** generate inner points from multiple (nested) polyhedra
- For any polyhedron $\mathcal{P}'_N \subset \mathcal{P}'$, we can also *heuristically* solve $\mathrm{OPT}\left(\mathcal{P}'_N, c^\top - u^\top A''\right)$
- Can greatly improve generation of incumbents, upper bounds on z_{IP}
- The more diverse the pool of columns, the better the chance to find good incumbents

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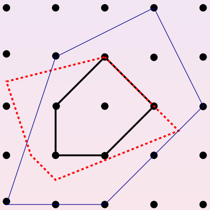

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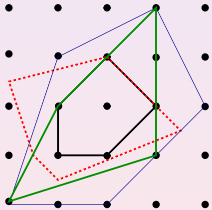

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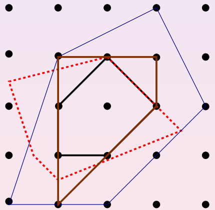

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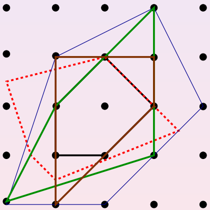

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Nested Polyhedra - Example

Vehicle Routing Problem

$$
\begin{array}{llll}
\min & \sum_{e \in E} c_e x_e & & \\
 & x(\delta(\{0\})) & = & 2k \\
 & x(\delta(\{v\})) & = & 2 & \forall v \in V \\
 & x(\delta(S)) & \geq & 2b(S) & \forall S \subseteq V,\ |S| > 1 \\
 & x_e \in \{0,1\} & & & \forall e \in E(V) \\
 & x_e \in \{0,1,2\} & & & \forall e \in \delta(\{0\})
\end{array}
$$

- **Relaxation:** b-Matching $\mathcal{P}' = \mathcal{P}^{\mathrm{bMatch}}$

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- **Nested Relaxation:** Multiple Traveling Salesman $\mathcal{P}^{\mathrm{kTSP}} \subset \mathcal{P}^{\mathrm{bMatch}}$

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- **Nested Relaxation:** b-Matching (plus some GSECs) $\mathcal{P}^{\mathrm{bMatch+}} \subset \mathcal{P}^{\mathrm{bMatch}}$

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Other Considerations

- **Integration of generic MILP cuts** - *new idea*
  - Using the mapping $\hat{x} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$ we can use generic MILP cuts in RC/PC context
- **Initial columns**
  - Solve $\mathrm{OPT}(\mathcal{P}', c + r)$ for random perturbations
  - Solve $\mathrm{OPT}(\mathcal{P}_N)$ heuristically
  - Run several iterations of LD or DC collecting extreme points
- **Price-and-branch heuristic**
  - For block-angular case, at end of each node, solve with $\lambda \in \mathbb{Z}$
  - Used in *root node* by Barahona and Jensen ('98), we extend to tree
- **Choice of master LP solver**
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  - Primal simplex after adding columns (warm-start primal feasible)
  - Interior-point methods might help with stabilization vs extremal duals
- **Compression of master LP and object pools**
  - Reduce size of master LP, improve efficiency of subproblem processing

Other Considerations

- **Integration of generic MILP cuts** - *new idea*
  - Using the mapping $\hat{x} = \sum_{s \in \mathcal{E}} s\hat{\lambda}_s$ we can use generic MILP cuts in RC/PC context
- **Initial columns**
  - Solve $\mathrm{OPT}(\mathcal{P}', c + r)$ for random perturbations
  - Solve $\mathrm{OPT}(\mathcal{P}_N)$ heuristically
  - Run several iterations of LD or DC collecting extreme points
- **Price-and-branch heuristic**
  - For block-angular case, at end of each node, solve with $\lambda \in \mathbb{Z}$
  - Used in *root node* by Barahona and Jensen ('98), we extend to tree
- **Choice of master LP solver**
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DIP Framework

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DIP (**D**ecomposition for **I**nteger **P**rogramming) is an open-source software framework that provides an implementation of various decomposition methods with minimal user responsibility

- Allows direct comparison CPM/DW/LD/PC/RC/DC in one framework
- DIP abstracts the common, generic elements of these methods
- Key: The user defines application-specific components in the space of the compact formulation - greatly simplifying the API
  - Define $[A'', b'']$ and/or $[A', b']$
  - Provide methods for $\mathrm{OPT}(\mathcal{P}', c)$ and/or $\mathrm{SEP}(\mathcal{P}', x)$
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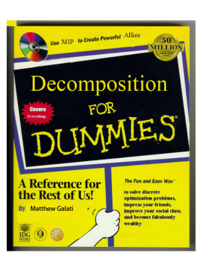

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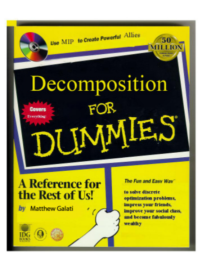

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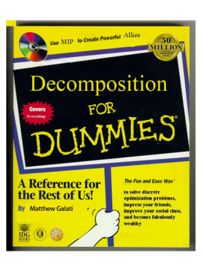

DIP Framework: Implementation

COmputational INfrastructure for Operations Research
Have some DIP with your CHiPPs?

- **DIP** was built around data structures and interfaces provided by COIN-OR
- The **DIP** framework, written in C++, is accessed through two user interfaces:
  - Applications Interface: `DecompApp`
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- **DIP** provides the bounding method for branch and bound
- **ALPS** (Abstract Library for Parallel Search) provides the framework for tree search
  - `AlpsDecompModel : public AlpsModel`
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- The base class `DecompApp` provides an interface for user to define the application-specific components of their algorithm
- Define the model(s)
  - `setModelObjective(double * c)`: define c
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  - `setModelRelaxed(DecompConstraintSet * model, int block)`: define $\mathcal{Q}'$ [optional]
- `solveRelaxed()`: define a method for $\mathrm{OPT}(\mathcal{P}', c)$ [optional, if $\mathcal{Q}'$, CBC is built-in]
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DIP Framework: Compare and Contrast to COIN/BCP

```
int main(int argc, char ** argv){
   //create the utility class for parsing parameters
   UtilParameters utilParam(argc, argv);
   bool doCut          = utilParam.GetSetting("doCut",       true);
   bool doPriceCut     = utilParam.GetSetting("doPriceCut",  false);
   bool doRelaxCut     = utilParam.GetSetting("doRelaxCut",  false);

   //create the user application (a DecompApp)
   SILP_DecompApp sip(utilParam);

   //create the CPM/PC/RC algorithm objects (a DecompAlgo)
   DecompAlgo    * algo = NULL;
   if(doCut)       algo = new DecompAlgoC (&sip, &utilParam);
   if(doPriceCut)  algo = new DecompAlgoPC(&sip, &utilParam);
   if(doRelaxCut)  algo = new DecompAlgoRC(&sip, &utilParam);

   //create the driver AlpsDecomp model
   AlpsDecompModel alpsModel(utilParam, algo);

   //solve
   alpsModel.solve();
}
```

DIP Framework: Compare and Contrast to COIN/BCP

- Limitations:
  - **BCP**: The user must derive methods for almost all of the algorithmic components: (master reformulation, expansion of rows and columns, branching in reformulated space, calculation of pricing mechanisms like reduced cost, etc).
  - **DIP**: There exists a compact formulation which forms the basis of the model attributes.
- Design:
  - **BCP**: The user defines the model attributes and algorithmic components based on one predefined solution *method* (i.e., PC or CPM).
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- Parallelization:
  - **BCP**: Designed for shared or distributed memory for branch-and-bound search.
  - **DIP**: Threaded for block-angular shared memory processing.
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- The base class DecompAlgo provides the shell (init / master / subproblem / update).
- Each of the methods described has derived default implementations DecompAlgoX : public DecompAlgo which are accessible by any application class, allowing full flexibility.
- New, hybrid or extended methods can be easily derived by overriding the various subroutines, which are called from the base class. For example,
  - Alternative methods for solving the master LP in DW, such as interior point methods
  - Add stabilization to the dual updates in LD (stability centers)
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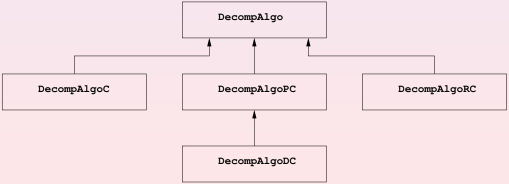

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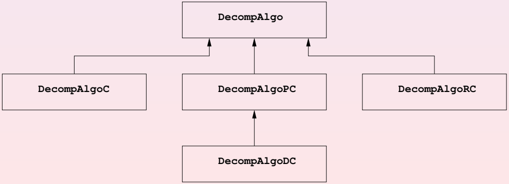

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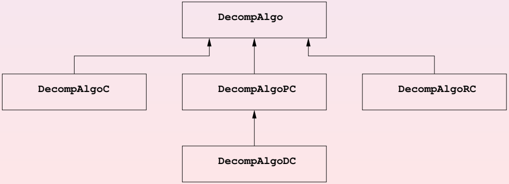

DIP - Example Applications

Application	Description	$\mathcal{P}'$	$\mathbf{OPT}(c)$	$\mathbf{SEP}(x)$	Input
AP3	3-index assignment	AP	Jonker	user	user
ATM	cash management (SAS COE)	MILP(s)	CBC	CGL	user
GAP	generalized assignment	KP(s)	Pisinger	CGL	user
MAD	matrix decomposition	MaxClique	Cliquer	CGL	user
MILP	random partition into A', A''	MILP	CBC	CGL	mps
MILPBlock	user-defined blocks for A'	MILP(s)	CBC	CGL	mps, block
MMKP	multi-dim/choice knapsack	MCKP	Pisinger	CGL	user
		MDKP	CBC	CGL	user
SILP	intro example, tiny IP	MILP	CBC	CGL	user
TSP	traveling salesman problem	1-Tree	Boost	Concorde	user
		2-Match	CBC	Concorde	user
VRP	vehicle routing problem	k-TSP	Concorde	CVRPSEP	user
		b-Match	CBC	CVRPSEP	user

Outline

Multi-Choice Multi-Dimensional Knapsack Problem (MMKP)

- **SAS Marketing Optimization** - improve ROI for marketing campaign offers by targeting higher response rates, improving channel effectiveness, and reduce spending.

$$
\begin{array}{llll}
\max & \sum_{i\in N}\sum_{j\in L_i} v_{ij}x_{ij} & & \\
& \sum_{i\in N}\sum_{j\in L_i} r_{kij}x_{ij} \leq b_k & & \forall k \in M \\
& \sum_{j\in L_i} x_{ij} = 1 & & \forall i \in N \\
& x_{ij} \in \{0,1\} & & \forall i \in N, j \in L_i
\end{array}
$$

- Relaxation - Multi-Choice Knapsack Problem (MCKP)
  - solver *mcknap* by Pisinger a DP-based branch-and-bound

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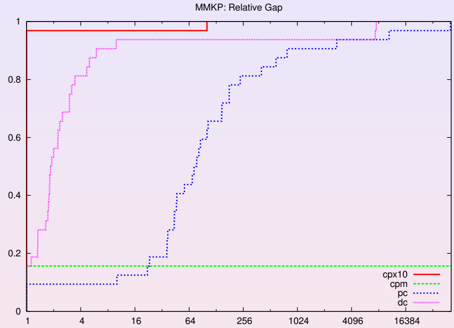
MMKP: CPX10.2 vs CPM/PC/DC

	CPX10.2		DIP-CPM		DIP-PC		DIP-DC	
Instance	**Time**	**Gap**	**Time**	**Gap**	**Time**	**Gap**	**Time**	**Gap**
I1	0.00	OPT	0.02	OPT	0.04	OPT	0.14	OPT
I10	T	0.05%	T	∞	T	11.86%	T	0.15%
I11	T	0.03%	T	∞	T	12.25%	T	0.14%
I12	T	0.01%	T	∞	T	7.93%	T	0.10%
I13	T	0.02%	T	∞	T	11.89%	T	0.12%
I2	0.01	OPT	0.01	OPT	0.05	OPT	0.05	OPT
I3	1.17	OPT	23.23	OPT	T	1.07%	T	0.75%
I4	15.71	OPT	T	∞	T	5.14%	T	0.77%
I5	0.01	0.01%	0.01	OPT	0.13	OPT	0.05	OPT
I6	0.14	OPT	0.07	OPT	T	0.28%	0.63	OPT
I7	T	0.08%	T	∞	T	14.32%	T	0.09%
I8	T	0.09%	T	∞	T	13.36%	T	0.20%
I9	T	0.06%	T	∞	T	10.71%	T	0.19%
INST01	T	0.43%	T	∞	T	9.99%	T	0.70%
INST02	T	0.09%	T	∞	T	7.39%	T	0.45%
INST03	T	0.38%	T	∞	T	3.83%	T	0.85%
INST04	T	0.34%	T	∞	T	7.48%	T	0.45%
INST05	T	0.18%	T	∞	T	10.23%	T	0.62%
INST06	T	0.21%	T	∞	T	9.82%	T	0.38%
INST07	T	0.36%	T	∞	T	15.75%	T	0.62%
INST08	T	0.25%	T	∞	T	11.55%	T	0.46%
INST09	T	0.21%	T	∞	T	15.24%	T	0.40%
INST11	T	0.22%	T	∞	T	7.96%	T	0.39%
INST12	T	0.18%	T	∞	T	7.90%	T	0.42%
INST13	T	0.08%	T	∞	T	2.97%	T	0.14%
INST14	T	0.05%	T	∞	T	3.89%	T	0.09%
INST15	T	0.04%	T	∞	T	3.43%	T	0.10%
INST16	T	0.06%	T	∞	T	2.19%	T	0.06%
INST17	T	0.03%	T	∞	T	2.09%	T	0.09%
INST18	T	0.03%	T	∞	T	4.43%	T	0.06%
INST19	T	0.03%	T	∞	T	3.13%	T	0.04%
INST20	T	0.03%	T	∞	T	3.05%	T	0.04%



	CPX10.2	DIP-CPM	DIP-PC	DIP-DC
Optimal	5	5	3	4
$\leq$ **1% Gap**	32	5	4	32
$\leq$ **10% Gap**	32	5	22	32

CGL: missing *Gub Covers*

MMKP: CPX10.2 vs CPM/PC/DC

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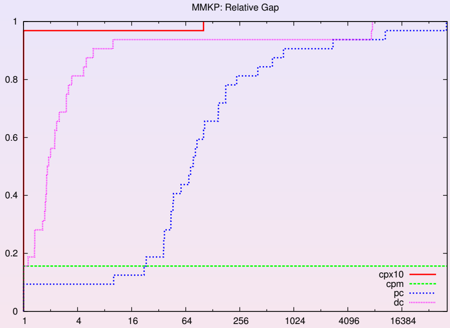
INST16	T	0.06%	T	∞	T	2.19%	T	0.06%
INST17	T	0.03%	T	∞	T	2.09%	T	0.09%
INST18	T	0.03%	T	∞	T	4.43%	T	0.06%
INST19	T	0.03%	T	∞	T	3.13%	T	0.04%
INST20	T	0.03%	T	∞	T	3.05%	T	0.04%



	CPX10.2	DIP-CPM	DIP-PC	DIP-DC
Optimal	5	5	3	4
$\leq$ **1% Gap**	32	5	4	32
$\leq$ **10% Gap**	32	5	22	32

CGL: missing *Gub Covers*

MMKP: Nested Pricing

- Nested Relaxations:
  - Multi-Choice 2-D Knapsack Problem (MC2KP): $\mathcal{P}_p^{\mathrm{MC2KP}} \subset \mathcal{P}^{\mathrm{MCKP}}\ \forall p \in M \setminus \{m\}$

$$\begin{array}{lcll}
\sum_{i \in N} \sum_{j \in L_i} r_{pij} x_{ij} & \leq & b_p & \\
\sum_{i \in N} \sum_{j \in L_i} r_{mij} x_{ij} & \leq & b_m & \\
\sum_{j \in L_i} x_{ij} & = & 1 & \forall i \in N \\
x_{ij} & \in & \{0,1\} & \forall i \in N, j \in L_i
\end{array}$$

  - Multi-Choice Multi-Dimensional Knapsack Problem (MMKP): $\mathcal{P} \subset \mathcal{P}^{\mathrm{MCKP}}$

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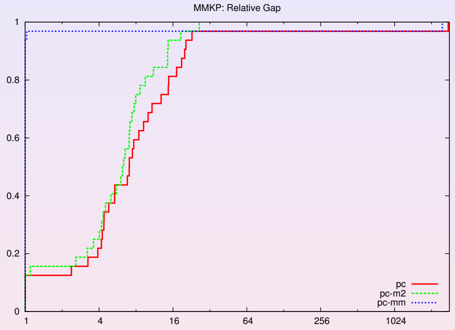
  - Multi-Choice Multi-Dimensional Knapsack Problem (MMKP): $\mathcal{P} \subset \mathcal{P}^{\text{MCKP}}$

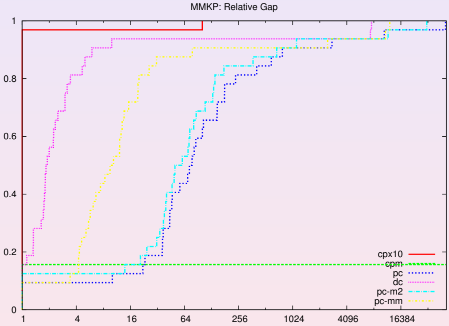
MMKP: PC vs PC Nested with MC2KP and MMKP

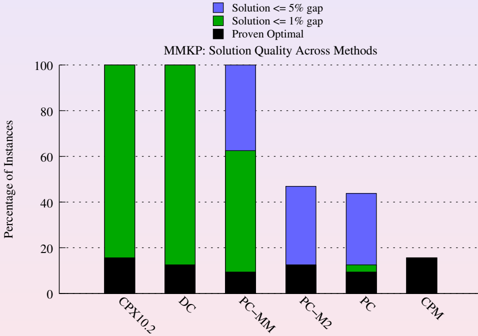
	DIP-PC		DIP-PC-M2		DIP-PC-MM	
Instance	**Time**	**Gap**	**Time**	**Gap**	**Time**	**Gap**
I1	0.04	OPT	0.16	OPT	0.08	OPT
I10	T	11.86%	T	6.99%	T	0.63%
I11	T	12.25%	T	11.15%	T	0.60%
I12	T	7.93%	T	11.41%	T	0.79%
I13	T	11.89%	T	13.65%	T	0.52%
I2	0.05	OPT	0.45	OPT	0.14	OPT
I3	T	1.07%	T	1.18%	T	1.10%
I4	T	5.14%	T	3.18%	T	1.23%
I5	0.13	OPT	0.14	OPT	0.07	OPT
I6	T	0.28%	483.53	OPT	T	0.25%
I7	T	14.32%	T	4.85%	T	0.97%
I8	T	13.36%	T	9.79%	T	0.67%
I9	T	10.71%	T	10.57%	T	0.73%
INST01	T	9.99%	T	5.97%	T	1.86%
INST02	T	7.39%	T	7.29%	T	1.74%
INST03	T	3.83%	T	11.93%	T	1.61%
INST04	T	7.48%	T	7.04%	T	1.56%
INST05	T	10.23%	T	8.84%	T	1.11%
INST06	T	9.82%	T	9.77%	T	1.39%
INST07	T	15.75%	T	8.78%	T	1.23%
INST08	T	11.55%	T	8.50%	T	1.37%
INST09	T	15.24%	T	8.48%	T	0.89%
INST11	T	7.96%	T	8.72%	T	1.13%
INST12	T	7.90%	T	6.72%	T	1.03%
INST13	T	2.97%	T	3.06%	T	0.76%
INST14	T	3.89%	T	3.67%	T	0.52%
INST15	T	3.43%	T	2.81%	T	0.78%
INST16	T	2.19%	T	3.01%	T	0.50%
INST17	T	2.09%	T	2.16%	T	0.39%
INST18	T	4.43%	T	2.60%	T	0.41%
INST19	T	3.13%	T	3.97%	T	0.46%
INST20	T	3.05%	T	4.06%	T	0.94%



	DIP-PC	DIP-PC-M2	DIP-PC-MM
Optimal	3	4	3
$\leq$ 1% Gap	4	4	20
$\leq$ 10% Gap	22	27	32

MMKP: CPX10.2 vs CPM/PC/DC/PC-M2/PC-MM





ATM Cash Management Problem - Business Problem

SAS Center of Excellence in Operations Research Applications (OR COE)

- Determine schedule for allocation of cash inventory at branch banks to service ATMs
- Define a polynomial fit for predicted cash flow need per day/ATM
- Predictive model factors include:
  - days of the week
  - weeks of the month
  - holidays
  - salary disbursement days
  - location of the branches
- Cash allocation plans finalized at beginning of month - deviations from plan are costly
- **Goal:** Determine multipliers for fit to minimize mismatch based on predicted withdrawals
- **Constraints:**
  - Regulatory agencies enforce a minimum cash reserve ratio at branch banks (per day)
  - For each ATM, limit on number of days *cash-out* based on predictive model (customer satisfaction)

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ATM Cash Management Problem - MINLP Formulation

- Simple *looking* nonconvex quadratic integer NLP.
- Linearize the absolute value, add binaries for count constraints.
- So far, no MINLP solvers seem to be able to solve this (several die with numerical failures).

$$
\begin{array}{lll}
\min \sum_{a \in A} \sum_{d \in D} |f_{ad}| & & \\
\text{s.t. } c^x_{ad} x_a + c^y_{ad} y_a + c^{xy}_{ad} x_a y_a + c^u_{ad} u_a + c_{ad} - w_{ad} & = f_{ad} & \forall a \in A, d \in D \\
\sum_{a \in A} (f_{ad} + w_{ad}) & \leq B_d & \forall d \in D \\
|\{d \in D \mid f_{ad} < 0\}| & \leq K_a & \forall a \in A \\
x_a, y_a & \in [0,1] & \forall a \in A \\
u_a & \geq 0 & \forall a \in A \\
f_{ad} & \geq -w_{ad} & \forall a \in A, d \in D
\end{array}
$$

Application - ATM Cash Management Problem - MILP Approx Formulation

- Discretization of x domain $\{0, 0.1, 0.2, ..., 1.0\}$.
- Linearization of product of binary and continuous, and absolute value.

$$
\begin{array}{llll}
\min & \sum_{a \in A} \sum_{d \in D} \left(f^+_{ad} + f^-_{ad} \right) & & \\
\text{s.t.} & c^x_{ad} \sum_{t \in T} c_t x_{at} + c^y_{ad} y_a + c^{xy}_{ad} \sum_{t \in T} c_t z_{at} + c^u_{ad} u_a - w_{ad} & = f^+_{ad} - f^-_{ad} & \forall a \in A, d \in D \\
& \sum_{t \in T} x_{at} & \leq 1 & \forall a \in A \\
& z_{at} & \leq x_{at} & \forall a \in A, t \in T \\
& z_{at} & \leq y_a & \forall a \in A, t \in T \\
& z_{at} & \geq x_{at} + y_a - 1 & \forall a \in A, t \in T \\
& f^-_{ad} & \leq w_{ad} v_{ad} & \forall a \in A, d \in D \\
& \sum_{a \in A} (f^+_{ad} - f^-_{ad} + w_{ad}) & \leq B_d & \forall d \in D \\
& \sum_{d \in D} v_{ad} & \leq K_a & \forall a \in A
\end{array}
$$

ATM Cash Management Problem - MILP Approx Formulation

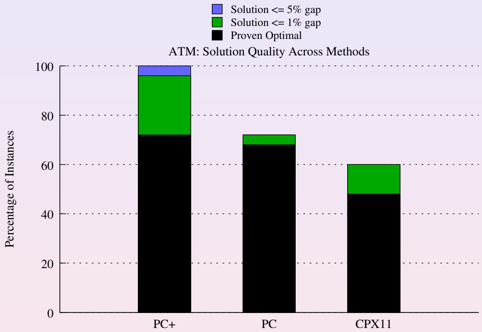
$$
\begin{array}{lll}
x_{at} & \in \{0,1\} & \forall a \in A, t \in T \\
z_{at} & \geq 0 & \forall a \in A, t \in T \\
v_{ad} & \in \{0,1\} & \forall a \in A, d \in D \\
y_{a} & \in [0,1] & \forall a \in A \\
u_{a} & \geq 0 & \forall a \in A \\
f^{+}_{ad}, f^{-}_{ad} & \in [0, w_{ad}] & \forall a \in A, d \in D
\end{array}
$$

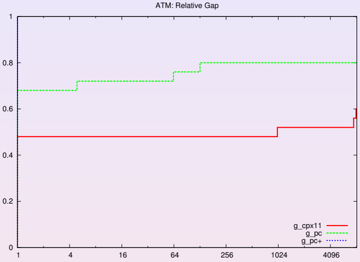
- The MILP formulation has a natural block-angular structure.
  - Master constraints are just the budget constraint.
  - Subproblem constraints (*the rest*) - one block for each ATM.

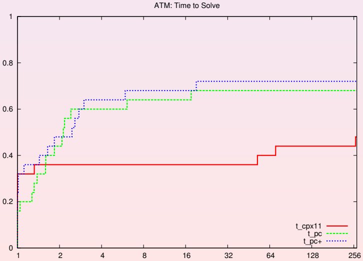
ATM: CPX11 vs PC/PC+

			CPX11			DIP-PC			DIP-PC+		
$\|A\|$	$\|D\|$	s	Time	Gap	Nodes	Time	Gap	Nodes	Time	Gap	Nodes
5	25	1	0.76	OPT	467	1.62	OPT	6	1.96	OPT	6
5	25	2	1.41	OPT	804	1.95	OPT	9	1.57	OPT	7
5	25	3	0.42	OPT	147	7.38	OPT	32	8.03	OPT	32
5	25	4	1.49	OPT	714	2.74	OPT	14	2.45	OPT	13
5	25	5	0.16	OPT	32	0.98	OPT	7	0.95	OPT	6
5	50	1	T	0.10	1264574	162.74	OPT	127	164.46	OPT	131
5	50	2	87.96	OPT	38341	183.28	OPT	273	263.24	OPT	275
5	50	3	8.09	OPT	3576	17.58	OPT	36	22.28	OPT	35
5	50	4	4.13	OPT	1317	3.13	OPT	3	3.17	OPT	3
5	50	5	57.55	OPT	32443	91.30	OPT	145	141.29	OPT	147
10	50	1	T	0.76	998624	297.65	OPT	301	234.47	OPT	156
10	50	2	1507.84	OPT	351879	28.84	OPT	29	52.99	OPT	29
10	50	3	T	0.81	667371	64.72	OPT	64	49.20	OPT	47
10	50	4	1319.00	OPT	433155	7.97	OPT	1	5.00	OPT	1
10	50	5	365.51	OPT	181013	12.49	OPT	3	5.18	OPT	3
10	100	1	T	∞	128155	T	∞	20590	T	0.11	13190
10	100	2	T	∞	116522	T	∞	60554	2437.43	OPT	135
10	100	3	T	∞	118617	T	∞	52902	T	0.20	40793
10	100	4	T	∞	108899	T	∞	47931	T	1.51	59477
10	100	5	T	∞	167617	T	∞	40283	T	0.38	26490
20	100	1	T	∞	93519	379.75	OPT	9	544.49	OPT	9
20	100	2	T	∞	68863	T	16.44	14240	T	0.26	25756
20	100	3	T	∞	95981	T	15.37	41495	T	0.12	3834
20	100	4	T	∞	81836	T	0.39	7554	T	0.08	7918
20	100	5	T	∞	101917	635.59	OPT	21	608.68	OPT	19
Optimal				12			17			18	
$\leq$ 1% Gap				15			18			25	
$\leq$ 10% Gap				15			18			25	

ATM: CPX11 vs PC/PC+







MILPBlock - Block-Angular MILP (as a Generic Solver)

- Consulting work led to numerous MILPs that cannot be solved with generic (B&C) solvers
- Often consider a decomposition approach, since a common modeling paradigm is
  - independent departmental policies which are then coupled by some global constraints
- Development time was slow due to problem-specific implementations of methods

$$\begin{pmatrix} A''_1 & A''_2 & \cdots & A''_\kappa \\ \hline A'_1 & & & \\ & A'_2 & & \\ & & \ddots & \\ & & & A'_\kappa \end{pmatrix}$$

- MILPBlock provides a black-box solver for applying integrated methods to generic MILP
  - This is the *first* framework to do this (to my knowledge).
  - Similar efforts are being talked about by F. Vanderbeck BaPCod (no cuts)
- Currently, the only input needed is MPS/LP and a *block file*
- Future work will attempt to embed automatic recognition of the block-angular structure using packages from linear algebra like: MONET, hMETIS, Mondriaan

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$$\left(\begin{array}{cccc} A_1'' & A_2'' & \cdots & A_\kappa'' \\ \hline A_1' & & & \\ & A_2' & & \\ & & \ddots & \\ & & & A_\kappa' \end{array}\right)$$

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Application - Block-Angular MILP (applied to Retail Optimization)

SAS Retail Optimization Solution

- *Multi-tiered supply chain distribution problem* where each block represents a store
- Prototype model developed in SAS/OR's OPTMODEL (algebraic modeling language)

	CPX11			DIP-PC		
Instance	**Time**	**Gap**	**Nodes**	**Time**	**Gap**	**Nodes**
retail27	T	2.30%	2674921	3.18	OPT	1
retail31	T	0.49%	1434931	767.36	OPT	41
retail3	529.77	OPT	2632157	0.54	OPT	1
retail4	T	1.61%	1606911	116.55	OPT	1
retail6	1.12	OPT	803	264.59	OPT	303

Outline

Future Research

- **Branch-and-Relax-and-Cut** - computational focus thus far has been on CPM/DC/PC
- Convergence issues and **stabilization** of duals (*stability centers*)
- Can we implement **Gomory cuts** in Price-and-Cut?
  - Similar to Interior Point crossover to Simplex, we can crossover from $\hat{x}$ to a feasible basis, load that into the solver and generate tableau cuts
  - Will the design of OSI and CGL work like this? **YES**. J Forrest has added a `crossover` to OsiClp
- Other generic MILP techniques for **MILPBlock**: heuristics, branching strategies, presolve
- DIP support for **identical subproblems** (using Vanderbeck's ideas)
- **Parallelization** of branch-and-bound
  - More work per node, communication overhead low - use ALPS
- **Parallelization** related to relaxed polyhedra (work-in-progress):
  - Pricing in block-angular case
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  - Traditional method for outer approximation: **cutting plane method**.
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  - Integrated methods: **price-and-cut** and **relax-and-cut**.
- *Introduction to a relatively new integrated method called* **decompose-and-cut**, *an associated class of cutting planes called* **decomposition cuts**, *and the concept of* **structured separation**.
- *Descriptions of numerous implementation considerations for branch-and-price-and-cut, including an introduction to a relatively unknown idea of using* **nested polyhedra** *for generating inner approximations.*
- **DIP** *(***D***ecomposition for* **I***nteger* **P***rogramming), an extensible open-source software framework for implementing decomposition-based methods with minimal user burden.*
- *MILPBlock, a DIP application and generic black-box solver for* **block-diagonal** *MILPs that fully automates the branch-and-price-and-cut algorithm with no additional user input.*
- *Computational results using DIP on three real-world* **applications** *coming from the marketing, banking, and retail industries.*

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