R-Linear Convergence of Limited Memory Steepest Descent

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joint work with

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Outline

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Unconstrained optimization: Steepest descent

Consider the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{where} \quad f : \mathbb{R}^n \to \mathbb{R} \quad \text{is} \quad \mathcal{C}^1.$$

Let us focus exclusively on a steepest descent framework:

Algorithm SD Steepest Descent

Require: $x_1 \in \mathbb{R}^n$

1: **for** $k \in \mathbb{N}$ **do**

2: Compute $g_k \leftarrow \nabla f(x_k)$

3: Choose $\alpha_k \in (0, \infty)$

4: Set $x_{k+1} \leftarrow x_k - \alpha_k g_k$

5: **end for**

All that remains to be determined are the stepsizes $\{\alpha_k\}$.

Minimizing strongly convex quadratics

Suppose $f(x) = \frac{1}{2}x^T Ax - b^T x$, where A has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$.

Convergence (rate) of the algorithm depends on choices for $\{\alpha_k\}$.

Minimizing strongly convex quadratics

Suppose $f(x) = \frac{1}{2}x^T Ax - b^T x$, where A has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$.

Choosing $\alpha_k \leftarrow 1/\lambda_n$ leads to Q-linear convergence with constant $(1 - \lambda_1/\lambda_n)$

Minimizing strongly convex quadratics

Suppose $f(x) = \frac{1}{2}x^T Ax - b^T x$, where A has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$.

…but certain "components" of the gradient vanish in a larger range.

Minimizing strongly convex quadratics

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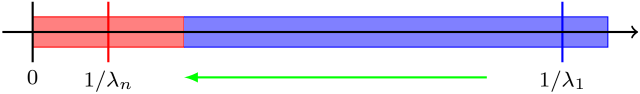

Goal: Allow large stepsizes, shrink range (automatically) to catch entire gradient.

Contributions

Consider Fletcher's limited memory steepest descent (LMSD) method.

- Extends the Barzilai-Borwein (BB) "two-point stepsize strategy".
- BB methods known to have R-linear convergence rate; Dai and Liao (2002).
- We prove that LMSD also attains R-linear convergence.

Although proved convergence rate is not necessarily better than that for BB, one can see reasons for improved empirical performance.

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Decomposition

$$\min_{x \in \mathbb{R}^n} f(x), \quad \text{where} \quad f(x) = \tfrac{1}{2} x^T A x - b^T x$$

Let A have the eigendecomposition $A = Q \Lambda Q^T$, where

$$\begin{aligned} Q &= \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} && \text{is orthogonal} \\ \text{and} \quad \Lambda &= \operatorname{diag}(\lambda_1, \dots, \lambda_n) && \text{with} \quad \lambda_n \geq \cdots \geq \lambda_1 > 0. \end{aligned}$$

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Let $g := \nabla f$. For any $x \in \mathbb{R}^n$, the gradient of f at x can be expressed as

$$g(x) = \sum_{i=1}^{n} d_i q_i, \quad \text{where} \quad d_i \in \mathbb{R} \quad \text{for all} \quad i \in [n] := \{1, \dots, n\}.$$

Recursion

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If $x^+ \leftarrow x - \alpha g(x)$, then the weights satisfy the recursive property:

$$d_i^+ = (1 - \alpha\lambda_i) d_i \quad \text{for all} \quad i \in [n].$$

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Proof (Sketch).

Since $g(x) = Ax - b$,

$$\begin{aligned} x^+ &= x - \alpha g(x) \\ Ax^+ &= Ax - \alpha g(x) \\ g(x^+) &= (I - \alpha A) g(x) \\ g(x^+) &= (I - \alpha Q \Lambda Q^T) g(x), \end{aligned}$$

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Idea: Choose stepsizes as reciprocals of (estimates of) eigenvalues of A.

LMSD method: Main idea

Fletcher (2012):

- Repeated cycles (or "sweeps") of m iterations.
- At start of $(k+1)$st cycle, suppose one has the kth cycle values in

$$G_k := \begin{bmatrix} g_{k,1} & \cdots & g_{k,m} \end{bmatrix} \text{ corresponding to } \{x_{k,1}, \dots, x_{k,m}\}.$$

- Iterate displacements lie in Krylov sequence initiated from $g_{k,1}$.

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- Iterate displacements lie in Krylov sequence initiated from $g_{k,1}$.
- Performing a QR decomposition to obtain

$$G_k = Q_k R_k,$$

one obtains m eigenvalue estimates (Ritz values) as eigenvalues of

$$\text{(symmetric tridiagonal)} \;\; T_k \leftarrow Q_k^T A Q_k,$$

which are contained in the spectrum of A in an optimal sense (more later).

- One can also obtain these estimates more cheaply and with less storage…

LMSD method: Efficient eigenvalue estimation

Storing the kth cycle reciprocal stepsizes in

$$J_k \leftarrow \begin{bmatrix} \alpha_{k,1}^{-1} & & \\ -\alpha_{k,1}^{-1} & \ddots & \\ & \ddots & \alpha_{k,m}^{-1} \\ & & -\alpha_{k,m}^{-1} \end{bmatrix},$$

one finds that by computing the (partially extended) Cholesky factorization

$$G_k^T \begin{bmatrix} G_k & g_{k,m+1} \end{bmatrix} = R_k^T \begin{bmatrix} R_k & r_k \end{bmatrix},$$

one has

$$T_k \leftarrow \begin{bmatrix} R_k & r_k \end{bmatrix} J_k R_k^{-1}.$$

Long story short: One can obtain Ritz values (and stepsizes) in $\sim \frac{1}{2}m^2n$ flops

- ... and this is done only once every m steps.

LMSD

Algorithm LMSD Limited Memory Steepest Descent

Require: $x_{1,1} \in \mathbb{R}^n$, $m \in \mathbb{N}$, and $\epsilon \in \mathbb{R}_+$

Choose stepsizes $\{\alpha_{1,j}\}_{j \in [m]} \subset \mathbb{R}_{++}$
Compute $g_{1,1} \leftarrow \nabla f(x_{1,1})$
if $\|g_{1,1}\| \leq \epsilon$, **then return** $x_{1,1}$
for $k \in \mathbb{N}$ **do**
    for $j \in [m]$ **do**
        Set $x_{k,j+1} \leftarrow x_{k,j} - \alpha_{k,j} g_{k,j}$
        Compute $g_{k,j+1} \leftarrow \nabla f(x_{k,j+1})$
        if $\|g_{k,j+1}\| \leq \epsilon$, **then return** $x_{k,j+1}$
    end for
    Set $x_{k+1,1} \leftarrow x_{k,m+1}$ and $g_{k+1,1} \leftarrow g_{k,m+1}$
    Set G_k and J_k
    Compute (R_k, r_k), then compute T_k
    Compute $\{\theta_{k,j}\}_{j \in [m]} \subset \mathbb{R}_{++}$ as the eigenvalues of T_k
    Compute $\{\alpha_{k+1,j}\}_{j \in [m]} \leftarrow \{\theta_{k,j}^{-1}\}_{j \in [m]} \subset \mathbb{R}_{++}$
end for

(Note: There is also a version using harmonic Ritz values.)

Known convergence properties

BB methods ($m = 1$):

- R-superlinear when $n = 2$; Barzilai and Borwein (1988)
- Convergent for any n from any starting point; Raydan (1993)
- R-linear for any n; Dai and Liao (2002)

LMSD methods ($m \geq 1$):

- Convergent for any n from any starting point; Fletcher (2012)
- Prior to our work: Convergence rate not yet analyzed.

Outline

Basic Assumptions

Assumption 1

(i) *Algorithm LMSD is run with* $\epsilon = 0$ *and* $g_{k,j} \neq 0$ *for all* $(k, j) \in \mathbb{N} \times [m]$.

(ii) *For all* $k \in \mathbb{N}$, *the matrix* G_k *has linearly independent columns. Further, there exists* $\rho \in [1, \infty)$ *such that, for all* $k \in \mathbb{N}$,

$$\|R_k^{-1}\| \leq \rho \|g_{k,1}\|^{-1}. \tag{2}$$

To justify (2), note that when $m = 1$, one has

$$Q_k R_k = G_k = g_{k,1} \quad \text{where} \quad Q_k = g_{k,1}/\|g_{k,1}\| \quad \text{and} \quad R_k = \|g_{k,1}\|.$$

Hence, (2) holds with $\rho = 1$.

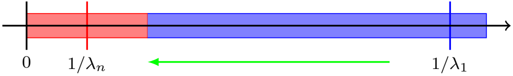
Intuition

Lemma 2

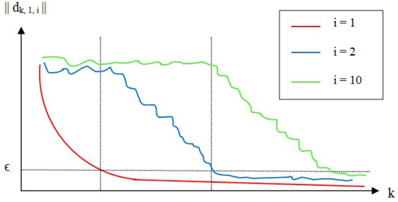
For all $k \in \mathbb{N}$, the eigenvalues of T_k satisfy

$$\theta_{k,j} \in [\lambda_{m+1-j}, \lambda_{n+1-j}] \subseteq [\lambda_1, \lambda_n] \quad \textit{for all} \quad j \in [m].$$

Recall. . .



We essentially prove that. . .



Worst-case "blow-up" of weights over a cycle

Lemma 3

For each $(k, j, i) \in \mathbb{N} \times [m] \times [n]$*:*

$$|d_{k,j+1,i}| \leq \delta_{j,i}|d_{k,j,i}| \quad \text{where} \quad \delta_{j,i} := \max\left\{\left|1 - \frac{\lambda_i}{\lambda_{m+1-j}}\right|, \left|1 - \frac{\lambda_i}{\lambda_{n+1-j}}\right|\right\}.$$

Hence, for each $(k, j, i) \in \mathbb{N} \times [m] \times [n]$*:*

$$|d_{k+1,j,i}| \leq \Delta_i|d_{k,j,i}| \quad \text{where} \quad \Delta_i := \prod_{j=1}^{m} \delta_{j,i}.$$

Furthermore, for each $(k, j, p) \in \mathbb{N} \times [m] \times [n]$*:*

$$\sqrt{\sum_{i=1}^{p} d_{k,j+1,i}^2} \leq \hat{\delta}_{j,p}\sqrt{\sum_{i=1}^{p} d_{k,j,i}^2} \quad \text{where} \quad \hat{\delta}_{j,p} := \max_{i \in [p]} \delta_{j,i},$$

while, for each $(k, j) \in \mathbb{N} \times [m]$*:*

$$\|g_{k+1,j}\| \leq \Delta\|g_{k,j}\| \quad \text{where} \quad \Delta := \max_{i \in [n]} \Delta_i.$$

Q-linear convergence of weight $i = 1$

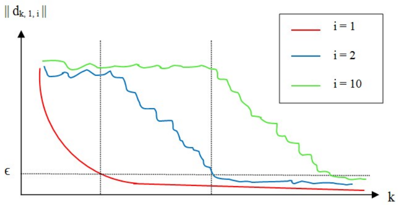
Lemma 4

If $\Delta_1 = 0$*, then* $d_{1+\hat{k},\hat{j},1} = 0$ *for all* $(\hat{k},\hat{j}) \in \mathbb{N} \times [m]$*. Otherwise, if* $\Delta_1 > 0$*, then:*

(i) *for* $(k,j) \in \mathbb{N} \times [m]$ *with* $d_{k,j,1} = 0$*, it follows that* $d_{k+\hat{k},\hat{j},1} = 0$ *for all* $(\hat{k},\hat{j}) \in \mathbb{N} \times [m]$*;*

(ii) *for* $(k,j) \in \mathbb{N} \times [m]$ *with* $|d_{k,j,1}| > 0$ *and any* $\epsilon_1 \in (0,1)$*, it follows that*

$$\frac{|d_{k+\hat{k},\hat{j},1}|}{|d_{k,j,1}|} \leq \epsilon_1 \quad \textit{for all} \quad \hat{k} \geq 1 + \left\lceil \frac{\log \epsilon_1}{\log \Delta_1} \right\rceil \quad \textit{and} \quad \hat{j} \in [m].$$



Ritz value representation

Lemma 5

For all $(k, j) \in \mathbb{N} \times [m]$, let $q_{k,j} \in \mathbb{R}^m$ denote the unit eigenvector corresponding to the eigenvalue $\theta_{k,j}$ of T_k, i.e., that with $T_k q_{k,j} = \theta_{k,j} q_{k,j}$ and $\|q_{k,j}\| = 1$. Then, defining

$$D_k := \begin{bmatrix} d_{k,1,1} & \cdots & d_{k,m,1} \\ \vdots & \ddots & \vdots \\ d_{k,1,n} & \cdots & d_{k,m,n} \end{bmatrix} \quad \text{and} \quad c_{k,j} := D_k R_k^{-1} q_{k,j},$$

it follows that, with the diagonal matrix of eigenvalues (namely, $\Lambda = Q^T A Q$),

$$\theta_{k,j} = c_{k,j}^T \Lambda c_{k,j} \quad \text{and} \quad c_{k,j}^T c_{k,j} = 1.$$

"If first p weights small, then bound for weight $p+1\dots$"

(We express $\hat{\delta}_p \in [1,\infty)$ dependent only on m, p, and the spectrum of A.)

Lemma 6 (simplified)

For any $(k,p) \in \mathbb{N} \times [n-1]$, if there exists $(\epsilon_p, K_p) \in (0, \frac{1}{2\hat{\delta}_p \rho}) \times \mathbb{N}$ with

$$\sum_{i=1}^{p} d_{k+\hat{k},1,i}^2 \leq \epsilon_p^2 \|g_{k,1}\|^2 \quad \text{for all} \quad \hat{k} \geq K_p,$$

then there exists $K_{p+1} \geq K_p$ dependent only on ϵ_p, ρ, and the spectrum of A with

$$d_{k+K_{p+1},1,p+1}^2 \leq 4\hat{\delta}_p^2 \rho^2 \epsilon_p^2 \|g_{k,1}\|^2;$$

Proof (Key step).

First p elements of $c_{k+\hat{k},j}$ small enough such that

$$\theta_{k+\hat{k},j} = \sum_{i=1}^{n} \lambda_i c_{k+\hat{k},j,i}^2 \geq \tfrac{3}{4}\lambda_{p+1} \quad \text{for} \quad \hat{k} \geq K_p \quad \text{and} \quad j \in [m].$$

“If first p weights small, then bound for all first $p+1$ weights. . . ”

Lemma 7

For any $(k,p) \in \mathbb{N} \times [n-1]$, *if there exists* $(\epsilon_p, K_p) \in (0, \frac{1}{2\hat{\delta}_p \rho}) \times \mathbb{N}$ *with*

$$\sum_{i=1}^{p} d_{k+\hat{k},1,i}^2 \leq \epsilon_p^2 \|g_{k,1}\|^2 \quad \textit{for all} \quad \hat{k} \geq K_p,$$

then, with $\epsilon_{p+1}^2 := (1 + 4\max\{1, \Delta_{p+1}^4\}\hat{\delta}_p^2 \rho^2)\epsilon_p^2$ *and* $K_{p+1} \in \mathbb{N}$,

$$\sum_{i=1}^{p+1} d_{k+\hat{k},1,i}^2 \leq \epsilon_{p+1}^2 \|g_{k,1}\|^2 \quad \textit{for all} \quad \hat{k} \geq K_{p+1}.$$

R-linear convergence of LMSD

Lemma 8

There exists $K \in \mathbb{N}$ *dependent only on the spectrum of* A *such that*

$$\|g_{k+K,1}\| \leq \tfrac{1}{2}\|g_{k,1}\| \quad \textit{for all} \quad k \in \mathbb{N}.$$

Theorem 9

The sequence $\{\|g_{k,1}\|\}$ *vanishes* R*-linearly in the sense that*

$$\|g_{k,1}\| \leq c_1 c_2^k \|g_{1,1}\|,$$

where

$$c_1 := 2\Delta^{K-1} \quad \textit{and} \quad c_2 := 2^{-1/K} \in (0,1).$$

Outline

Numerical demonstrations with $n = 100$: $m = 1$ and $m = 5$

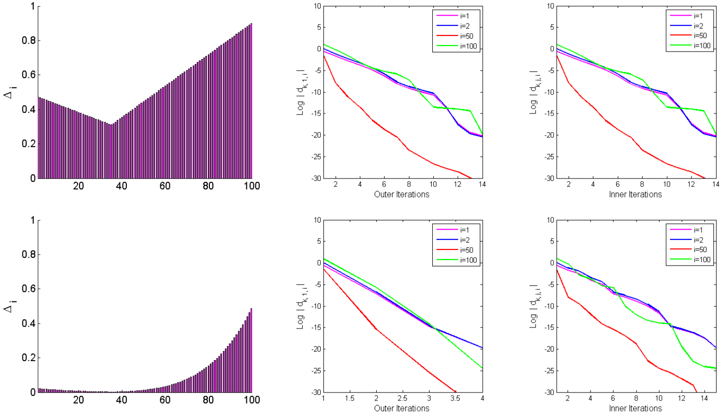


Figure: $\{\lambda_1, \ldots, \lambda_{100}\} \subset [1, 1.9]$

Numerical demonstrations with $n = 100$: $m = 1$ and $m = 5$

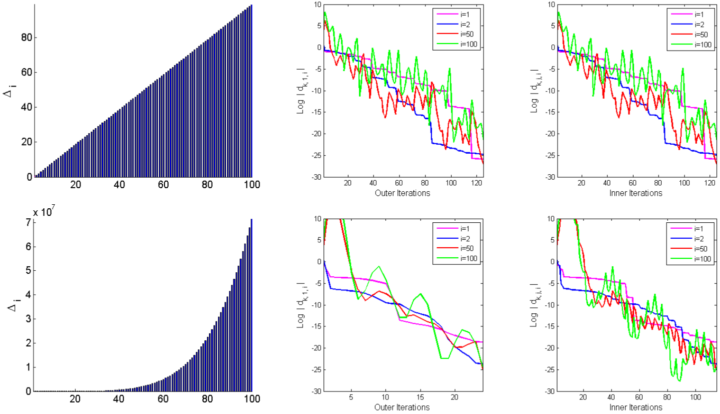


Figure: $\{\lambda_1, \ldots, \lambda_{100}\} \subset [1, 100]$

Numerical demonstrations with $n = 100$: $m = 1$ and $m = 5$

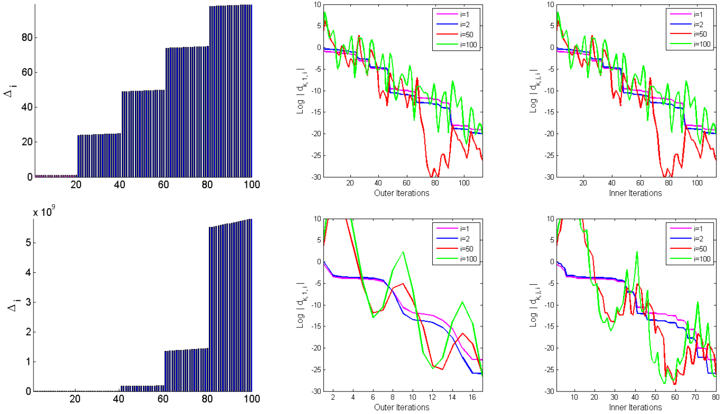


Figure: $\{\lambda_1, \ldots, \lambda_{100}\} \subset 5$ clusters, $m = 5$

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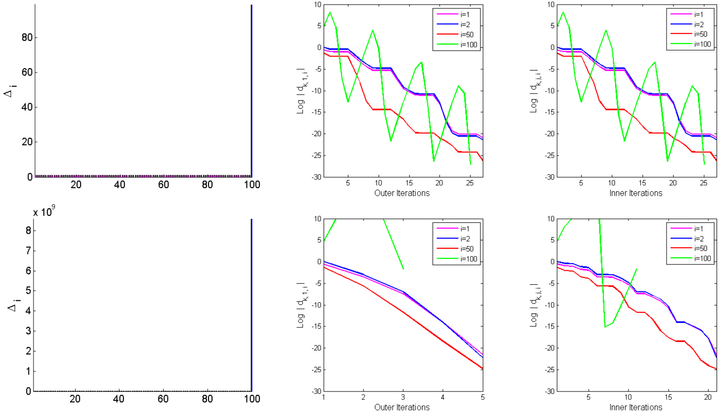

Figure: $\{\lambda_1, \ldots, \lambda_{100}\} \subset 2$ clusters (low heavy), $m = 5$

Numerical demonstrations with $n = 100$: $m = 1$ and $m = 5$

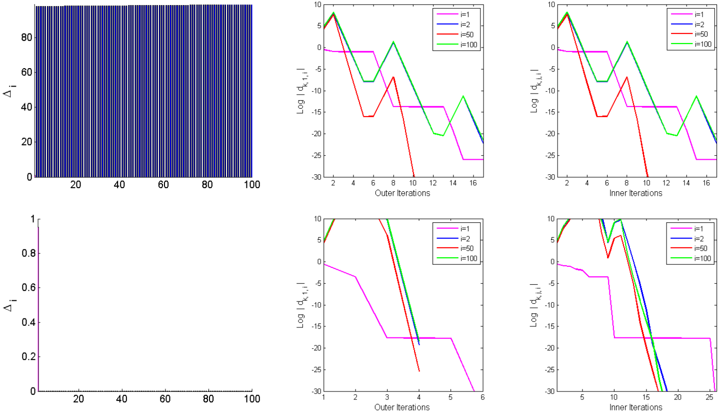

Figure: $\{\lambda_1, \ldots, \lambda_{100}\} \subset 2$ clusters (high heavy), $m = 5$

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⋆ F. E. Curtis and W. Guo.
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Technical Report 16T-010, COR@L Laboratory, Department of ISE, Lehigh University, 2016.

Soon in *IMA Journal of Numerical Analysis*: https://doi.org/10.1093/imanum/drx016