Infeasibility Detection in Nonlinear Optimization

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(See also "Infeasibility Detection in Nonlinear Programming," Figen Oztoprak, MS71)

Outline

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Motivation

Penalty-SQO Method

Penalty-Interior-Point Method

Unified Framework(?)

Constrained minimization

Is this how we should formulate optimization problems?

$$\min_{x} f(x) \quad \text{s.t.} \begin{cases} c^{\mathcal{E}}(x) = 0 \\ c^{\mathcal{I}}(x) \leq 0 \end{cases} \tag{OP}$$

Feasibility violation minimization

Suppose that if (OP) is infeasible, then we want to solve

$$\min_x \; v(x) := \left\| \begin{bmatrix} c^{\mathcal{E}}(x) \\ \max\{c^{\mathcal{I}}(x), 0\} \end{bmatrix} \right\| \tag{FP}$$

Many algorithms/codes do this already, by either

- switching from solving one problem to the other;
- transitioning from solving one problem to the other.

But are they doing it efficiently?

Numerical experiments: Infeasible optimization problems

Iterations and f evaluations for 8 infeasible optimization problems (2-3 variables):

Prob.	Ipopt		Knitro-Direct		Knitro-Active		Filter	
	Iter.	Eval.	Iter.	Eval.	Iter.	Eval.	Iter.	Eval.
1	48	281	38	135	22	235	16	16
2	109	170	*10000	*40544	23	167	12	12
3	788	3129	12	83	9	202	10	10
4	46	105	25	61	10	201	11	11
5	72	266	*1060	*3401	18	45	26	26
6	63	141	*76	*264	16	37	27	27
7	87	152	*10000	*43652	*10000	*20091	30	30
8	104	206	33	97	41	560	28	28

Problems also run with `Knitro-CG` and `LOQO`, but they failed every time.

(We want your infeasible test problems!)

Numerical experiments: Feasibility problems (solved directly)

Iterations and f evaluations for 8 feasibility problems (2-3 variables):

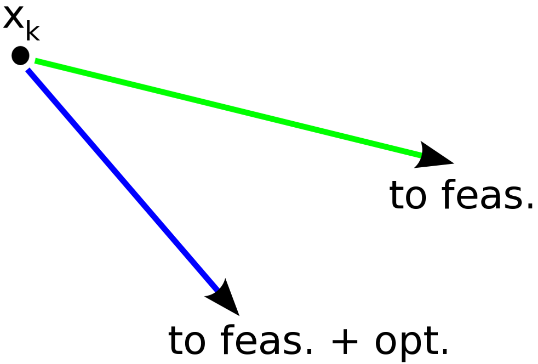
Problem	Ipopt		Knitro-Direct		Knitro-Active		Filter	
	Iter.	Eval.	Iter.	Eval.	Iter.	Eval.	Iter.	Eval.
1	28	29	14	15	13	24	17	21
2	31	32	31	33	8	9	12	13
3	50	131	10	11	9	12	12	13
4	24	79	18	29	10	13	10	12
5	166	786	29	40	21	24	30	32
6	37	48	20	21	19	22	26	27
7	59	65	31	34	19	20	25	28
8	46	71	19	20	23	28	26	29

$\Longrightarrow$ If we can switch/transition efficiently, then our current tools work well.

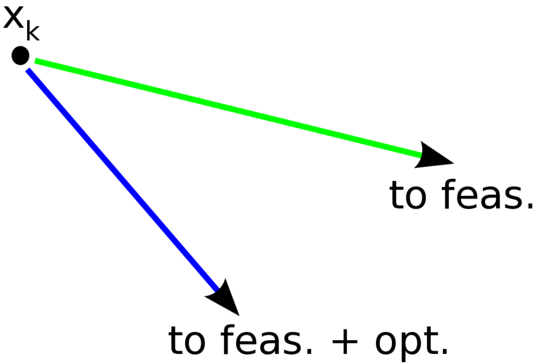
Rapid detection of infeasibility

Problem type	Global convergence	Fast local convergence
Feasible	✓	✓
Infeasible	✓	?

Two-phase strategy



Two-phase strategy



- Exploratory step determines possible progress toward feasibility.
- Rapid infeasibility detection requires rapid transition/switch to feasibility steps.
- (Important to have consistency between ways the two steps measure feasibility.)

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Penalty methods

Penalization provides one mechanism for transitioning to infeasibility detection:

$$\min_{x,r,s,t} \rho f(x) + e^T r + e^T s + e^T t$$
$$\text{s.t.} \begin{cases} c^{\mathcal{E}}(x) = r - s \\ c^{\mathcal{I}}(x) \leq t \\ (r, s, t) \geq 0 \end{cases} \qquad \text{(PP)}$$

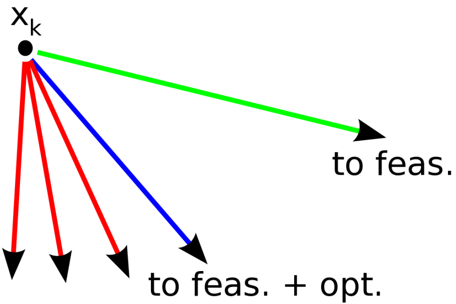
How should ρ be updated?

Literature

- (Rich history of SQP methods)
- Fletcher, Leyffer (2002)
- Byrd, Gould, Nocedal (2005)
- Byrd, Nocedal, Waltz (2008)
- Byrd, Curtis, Nocedal (2010)
- Byrd, López-Calva, Nocedal (2010)
- Gould, Robinson (2010)
- Morales, Nocedal, Wu (2010)

Steering and FILTER strategies

- Steering methods solve a sequence of constrained subproblems:



- FILTER: "... we make use of a property of the phase I algorithm in our QP solver. If an infeasible QP is detected, the solver exists with a solution of [an LP minimizing the ℓ_1 norm of violated constraints]."

SQuID strategy

- Compute v_k as the optimal value for

$$\min_{d,r,s,t} \; e^T r + e^T s + e^T t$$
$$\text{s.t.} \begin{cases} c^{\mathcal{E}}(x_k) + \nabla c^{\mathcal{E}}(x_k)^T d = & r - s \\ c^{\mathcal{I}}(x_k) + \nabla c^{\mathcal{I}}(x_k)^T d \leq & t \\ \|d\|_\infty \leq & \Delta \\ (r,s,t) \geq & 0 \end{cases}$$

- Update ρ for rapid infeasibility detection (Byrd, Curtis, Nocedal, 2010)
- Compute d_k as the search direction from

$$\min_{d,r,s,t} \; \rho(f(x_k) + \nabla f(x_k)^T d) + e^T r + e^T s + e^T t + \tfrac{1}{2} d^T H(x_k, \lambda_k; \rho) d$$
$$\text{s.t.} \begin{cases} c^{\mathcal{E}}(x_k) + \nabla c^{\mathcal{E}}(x_k)^T d = & r - s \\ c^{\mathcal{I}}(x_k) + \nabla c^{\mathcal{I}}(x_k)^T d \leq & t \\ e^T r + e^T s + e^T t \leq & v_k \\ (r,s,t) \geq & 0 \end{cases}$$

- Update ρ for descent

Numerical experiments: Infeasible optimization problems

Iterations and f evaluations for 8 infeasible optimization problems (2-3 variables):

Prob.	Filter		SQuID	
	Iter.	Eval.	Iter.	Eval.
1	16	16	32	65
2	12	12	12	18
3	10	10	14	36
4	11	11	26	57
5	26	26	5	16
6	27	27	18	18
7	30	30	19	19
8	28	28	17	17

Outline

Penalty and interior-point methods

Constrained subproblems in penalty methods can be expensive:

$$\min_{x,r,s,t} \rho f(x) + e^T r + e^T s + e^T t \quad \text{s.t.} \begin{cases} c^{\mathcal{E}}(x) = r - s \\ c^{\mathcal{I}}(x) \leq t \\ (r,s,t) \geq 0 \end{cases} \tag{PP}$$

Interior-point methods are more efficient for large-scale problems:

$$\min_{x,u} f(x) - \mu \sum \ln u^i \quad \text{s.t.} \begin{cases} c^{\mathcal{E}}(x) = 0 \\ c^{\mathcal{I}}(x) = -u \\ u \geq 0 \end{cases} \tag{IP}$$

Penalty-interior-point method

Applying an interior-point reformulation to (PP):

$$\min_{x,r,s,t,u} \rho f(x) - \mu \left(\sum (\ln r^i + \ln s^i) + \sum (\ln t^i + \ln u^i) \right) + e^T r + e^T s + e^T t$$
$$\text{s.t.} \begin{cases} c^{\mathcal{E}}(x) = & r - s \\ c^{\mathcal{I}}(x) = & t - u \end{cases} \tag{PIP}$$

The optimization problem (OP) and feasibility problem (FP) can be solved via (PIP):

- $\mu \to 0$ and $\rho \to \bar{\rho} > 0$ to solve (OP).
- $\mu \to 0$ and $\rho \to 0$ to solve (FP).

Literature

Previous work with similar motivations:

- Jittorntrum and Osborne (1980)
- Polyak (1982, 1992, 2008)
- Breitfeld and Shanno (1994, 1996)
- Goldfarb, Polyak, Scheinberg, and Yuzefovich (1999)
- Gould, Orban, and Toint (2003)
- Chen and Goldfarb (2006, 2006)
- Benson, Sen, and Shanno (2008)

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- Benson, Sen, and Shanno (2008)

Parameter updates are essential to have a practical algorithm.

Conservative strategy:

1. Fix $\rho > 0$ and (approximately) solve (PIP) for $\mu \to 0$.
2. If infeasible, then reduce ρ and go to step 1.

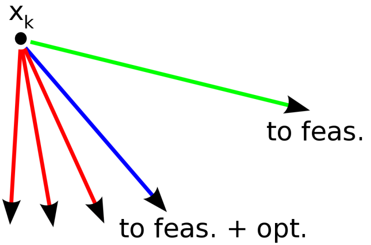
(ρ reduced too slowly to ensure fast convergence for infeasible problems.)

PIPAL strategy

Newton system has the form

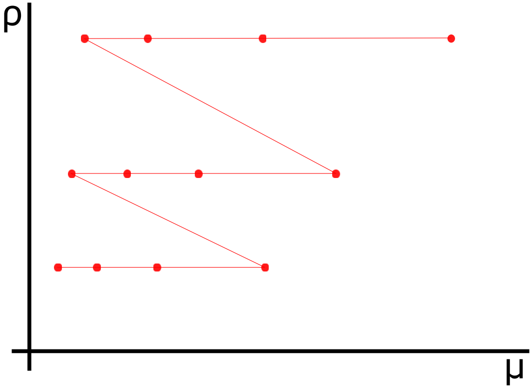
$$A_{\rho,\mu} d = \rho q_1 + \mu q_2 + q_3.$$

Steering strategy now efficient! (Only one factorization required.)



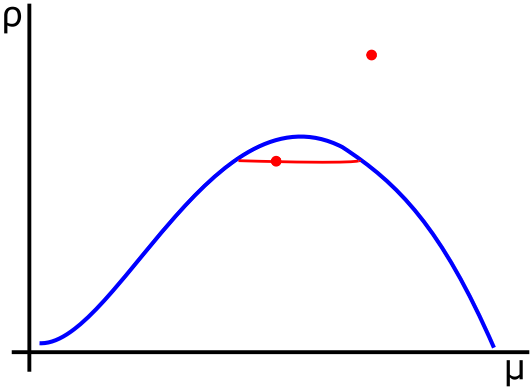
- Solution for $(\rho, \mu) = (0, \mu)$ indicates progress possible toward primal feasibility.
- Adjust ρ to aim for primal feasibility.
- Adjust μ to aim for dual feasibility and complementarity.

Conservative strategy



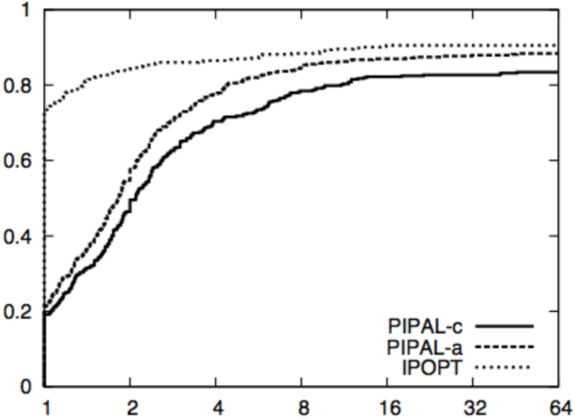
- Each “dot” may require at least a few iterations.
- Each “row” may require the computational effort of an entire interior-point solve.
- For an infeasible problem, we need both ρ and μ to reduce to zero.

PIPAL strategy

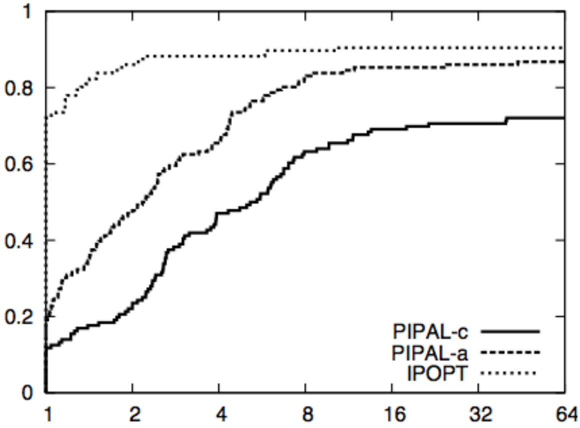


- Blue line separates acceptable and unacceptable ρ values (for primal feasibility).
- μ then set to ensure progress toward dual feasibility and complementarity.
- For an infeasible problem, ρ and μ can reduce rapidly.

Numerical results: Feasible problems (sample size = 422)

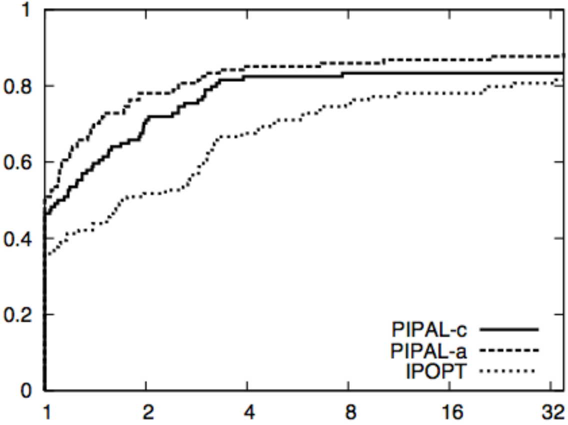


Numerical results: Feasible problems w/ ρ decrease (sample size = 136)



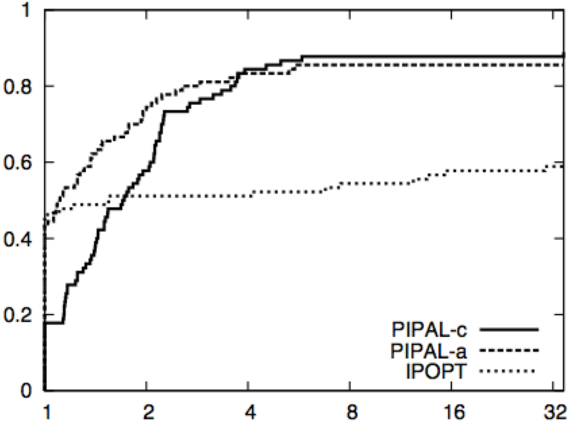
Numerical results: Degenerate problems (sample size = 114)

Added constraints: $c^i(x)^2 \geq 0$



Numerical results: Infeasible problems (sample size = 90)

Added constraints: $c^i(x)^2 \leq -1$



Outline

Constrained minimization

Is this how we should formulate optimization problems?

$$\min_x \; f(x) \quad \text{s.t.} \begin{cases} c^{\mathcal{E}}(x) = 0 \\ c^{\mathcal{I}}(x) \leq 0 \end{cases} \tag{OP}$$

That is, should the strategy be the following:

- Attempt to solve (OP), but...
- if it is infeasible, then solve

$$\min_x \; v(x) := \left\| \begin{bmatrix} c^{\mathcal{E}}(x) \\ \max\{c^{\mathcal{I}}(x), 0\} \end{bmatrix} \right\| \tag{FP}$$

Minimization over set of feasibility violation minimizers

Let $\mathcal{X}$ be the solution set of feasibility violation minimization:

$$\min_x \; v(x) := \left\| \begin{bmatrix} c^{\mathcal{E}}(x) \\ \max\{c^{\mathcal{I}}(x), 0\} \end{bmatrix} \right\| \tag{FP}$$

We should solve

$$\begin{aligned} \min_x \; & f(x) \\ \text{s.t. } & x \in \mathcal{X}. \end{aligned} \tag{UP}$$

(Even better would be a single problem or set of conditions for solving (UP).)

$1 \leq 0$

Consider the following model (AMPL presolver turned off):

```
var      x;
minimize f: (x-1)^2;
s.t.     c: 1 <= 0;
```

Solver	Inf. Declared?	Iterations	Final (x, y)
CONOPT	No	0	0.0000
FILTER	Yes	0	0.0000
IPOPT	Yes	6	0.0099
KNITRO	Yes	7	0.6667
LANCELOT	Yes	1	1.0000
LOQO	No	500	0.6728
MINOS	Yes	0	0.0000
MOSEK	Yes	1	1.0000
PENNON	No	0	1.0000
SNOPT	Yes	0	0.0000

$y^2 \leq -1$

Consider the following model (AMPL presolver turned off):

```
var      x;
var      y;
minimize f: (x-1)^2;
s.t.     c: y^2 + 1 <= 0;
```

Solver	Inf. Declared?	Iterations	Final (x, y)
CONOPT	Yes	3	(0.0000,0.0000)
FILTER	Yes	2	(0.0000,0.0000)
IPOPT	Yes	6	(0.0099,0.0000)
KNITRO	Yes	4	(0.3765,0.0000)
LANCELOT	Yes	1	(1.0000,0.0000)
LOQO	No	500	(0.4569,0.0000)
MINOS	Yes	2	(1.0000,0.0000)
MOSEK	Yes	1	(0.0000,0.0000)
PENNON	No	0	(1.0000,0.0000)
SNOPT	Yes	3	(1.0000,0.0000)

Penalty and switching methods

- Penalization provides one mechanism for transitioning to infeasibility detection:

$$\min_{x,r,s,t} \ \rho f(x) + e^T r + e^T s + e^T t$$
$$\text{s.t.} \begin{cases} c^{\mathcal{E}}(x) = & r - s \\ c^{\mathcal{I}}(x) \leq & t \\ (r,s,t) \geq & 0 \end{cases} \tag{PP}$$

However, once $x \in \mathcal{X}$ is found, $f(x)$ has been thrown out (since $\rho \to 0$).

- Switching methods throw out $f(x)$ even earlier.

What is needed?

General-purpose algorithms that:

- require no constraint qualifications to do something useful
- minimize the objective over the set of feasibility violation minimizers
- are as efficient and robust as contemporary methods when problems are nice