A Sequential Algorithm for Solving Nonlinear Optimization Problems With Chance Constraints

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joint work with

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Outline

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Chance-constrained optimization

$$\min_{x \in \mathbb{R}^n} f(x)$$
$$\text{s.t. } \mathbb{P}_\xi[c(x,\xi) \leq 0] \geq 1 - \alpha$$

$f : \mathbb{R}^n \to \mathbb{R}$

$c : \mathbb{R}^n \times \Omega \to \mathbb{R}^m$

- Random variable ξ with associated space $(\Omega, \mathcal{F}, P)$
- Assume f and $c(\cdot, \xi)$ are $\mathcal{C}^1$ for any realization of ξ
- Even if f and c linear, the feasible region is nonconvex
- Assume $m = 1$ (though $m > 1$ or multiple chance constraints also OK)

Cardinality-constrained optimization

- Sample Average Approximation (SAA):

$$\Omega = \{\xi_1, \ldots, \xi_N\} \text{ with equal probability}$$

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & f(x) \\ \text{s.t. } & |\{\xi_i \in \Omega : c(x, \xi_i) \leq 0\}| \geq \lceil (1-\alpha)N \rceil \end{aligned} \tag{CCP}$$

Cardinality-constrained optimization

- Sample Average Approximation (SAA):

$$\Omega = \{\xi_1, \ldots, \xi_N\} \text{ with equal probability}$$

$$\boxed{\begin{aligned} \min_{x \in \mathbb{R}^n} \; & f(x) \\ \text{s.t. } & |\{c_i(x) \leq 0\}| \geq M \end{aligned}} \qquad \text{(CCP)}$$

…arise in other applications as well

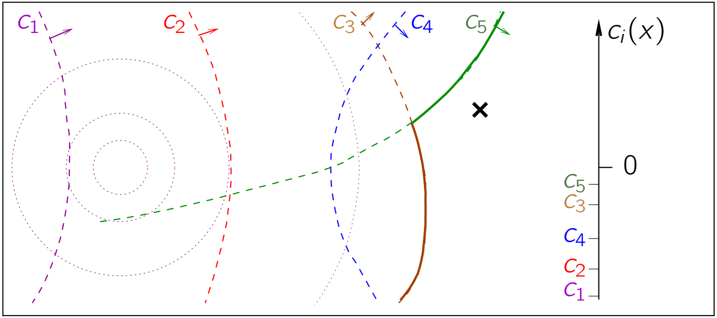
Contribution

Propose an SQP-type method for solving problem CCP

- Viable approach even when, e.g., MINLP techniques are intractable
- Like penalty-SQP, sequential minimization of penalty function
- Novel penalty function with “exactness” properties
- Subproblems with linear chance constraints
  - Do not need to “predict” optimal subset of constraints
  - Efficient methods, e.g. [Luedtke 14], [Kücükyavuz 12]
- Global convergence guarantees
- Good performance on test problems

Outline

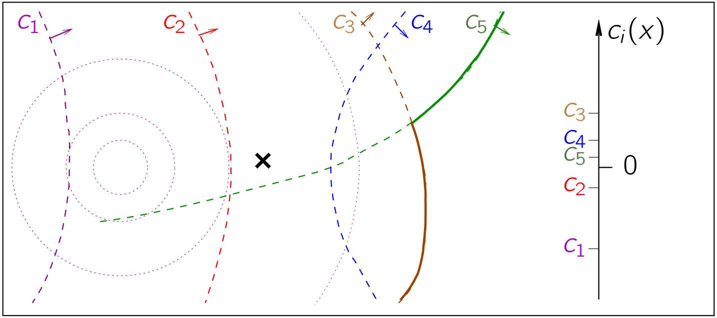
Exact penalty function for deterministic optimization



$$
\begin{aligned}
\phi(x) &= f(x) + \rho \, \| [c(x)]_+ \|_1 \\
&= f(x) + \rho \sum_{i=1}^{N} \max\{0, c_i(x)\}
\end{aligned}
$$

Under CQ, local mins of ϕ correspond to local mins of optimization problem

Exact penalty function for deterministic optimization



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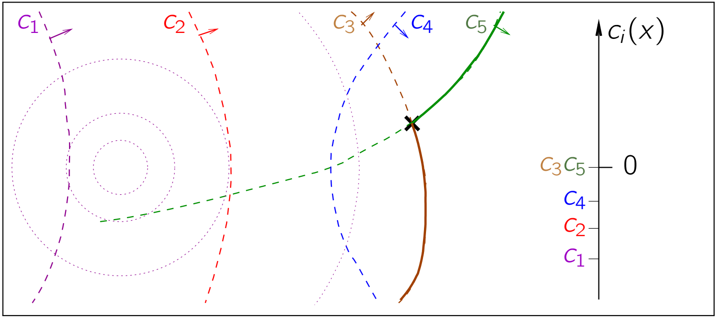

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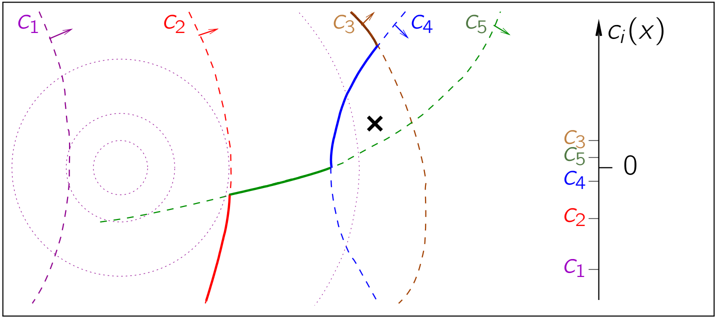
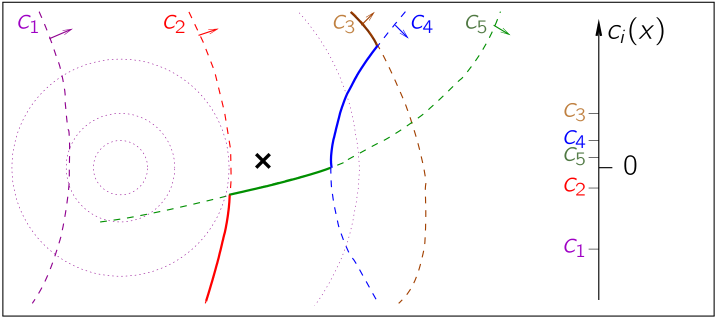
Under CQ, local mins of ϕ correspond to local mins of optimization problem

Constraint violation measure for CCP



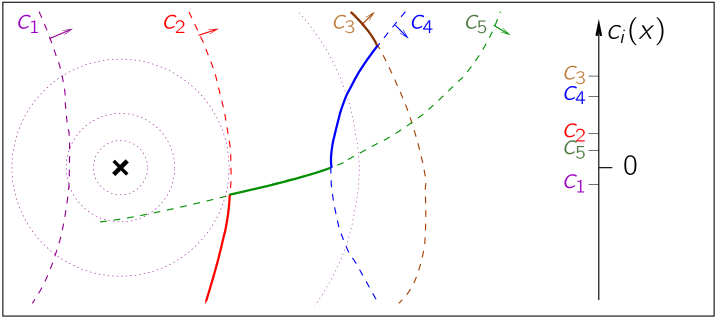
- Order constraints: $c_{i(1)}(x) \leq c_{i(2)}(x) \leq \ldots$
- Violation measure: $\langle\langle [c(x)]_+ \rangle\rangle_M = \sum_{k=1}^{M} \max\{0, c_{i(k)}(x)\}$
- Here: $\langle\langle [c(x)]_+ \rangle\rangle_M = 0$

Constraint violation measure for CCP



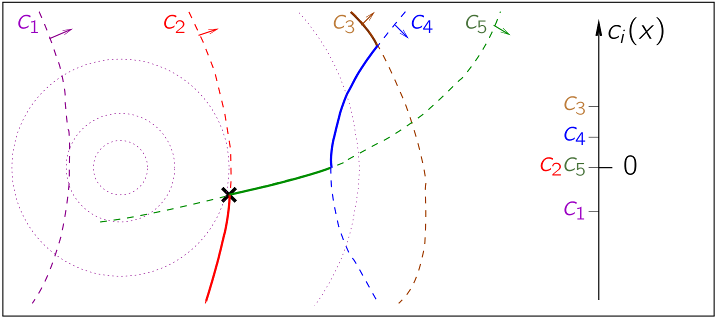
- Order constraints: $c_{i(1)}(x) \leq c_{i(2)}(x) \leq \dots$
- Violation measure: $\langle\langle [c(x)]_+ \rangle\rangle_M = \sum_{k=1}^{M} \max\{0, c_{i(k)}(x)\}$
- Here: $\langle\langle [c(x)]_+ \rangle\rangle_M = c_5(x)$

Constraint violation measure for CCP



- Order constraints: $c_{i(1)}(x) \leq c_{i(2)}(x) \leq \dots$
- Violation measure: $\langle\langle [c(x)]_+ \rangle\rangle_M = \sum_{k=1}^{M} \max\{0, c_{i(k)}(x)\}$
- Here: $\langle\langle [c(x)]_+ \rangle\rangle_M = c_2(x) + c_5(x)$

Constraint violation measure for CCP



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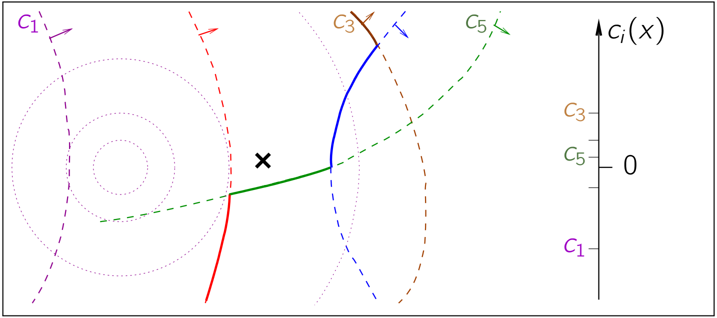
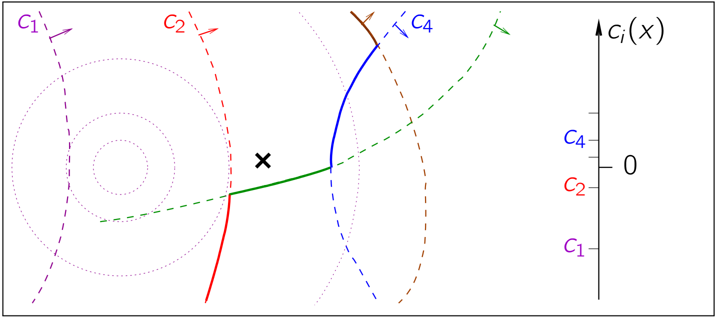
Equivalent formulation



$$\mathfrak{S} := \left\{\mathcal{S} \subseteq \{1,\ldots,N\} : |\mathcal{S}| = M\right\} \qquad c_{\mathcal{S}}(x) := [c_i(x)]_{i\in\mathcal{S}}$$

- Consider $\|[c_{\mathcal{S}}(x)]_+\|_1$ for different $\mathcal{S} \in \mathfrak{S}$
- Here: $\mathcal{S} = \{1, 3, 5\}$: $\|[c_{\mathcal{S}}(x)]_+\|_1 = c_5(x) + c_3(x)$

Equivalent formulation



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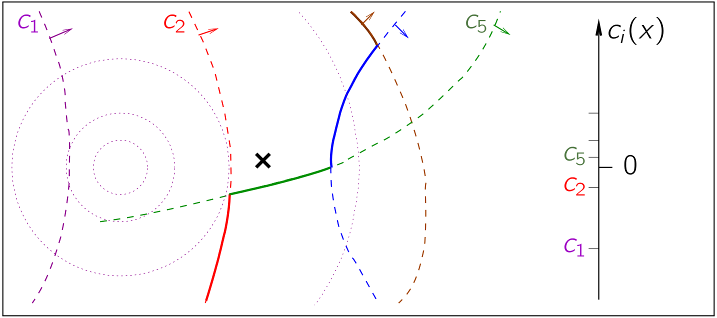
- Consider $\|[c_{\mathcal{S}}(x)]_+\|_1$ for different $\mathcal{S} \in \mathfrak{S}$
- Here: $\mathcal{S} = \{1, 2, 4\}$: $\|[c_{\mathcal{S}}(x)]_+\|_1 = c_4(x)$

Equivalent formulation



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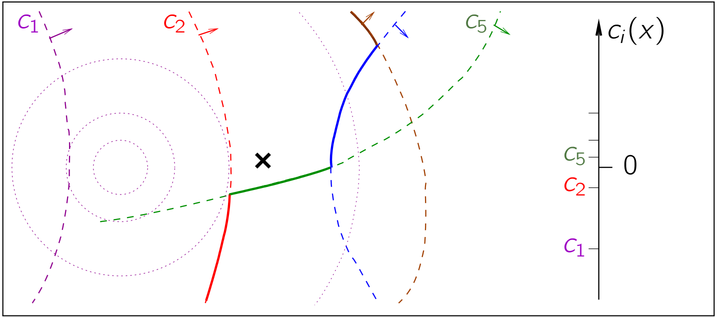
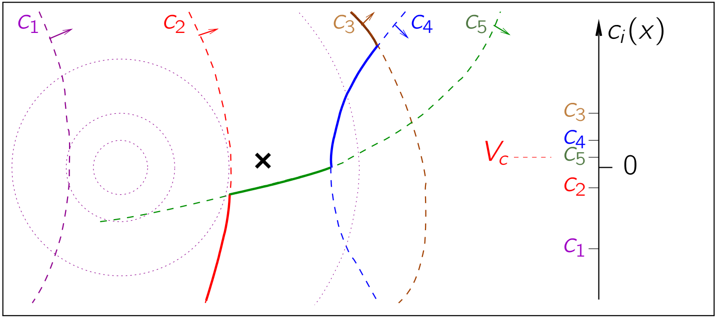
Equivalent formulation



$$\mathfrak{S} := \left\{\mathcal{S} \subseteq \{1, \ldots, N\} : |\mathcal{S}| = M\right\} \qquad c_{\mathcal{S}}(x) := [c_i(x)]_{i \in \mathcal{S}}$$

- Consider $\|[c_{\mathcal{S}}(x)]_+\|_1$ for different $\mathcal{S} \in \mathfrak{S}$
- So $\langle\!\langle [c(x)]_+ \rangle\!\rangle_M = \min_{\mathcal{S} \in \mathfrak{S}} \|[c_{\mathcal{S}}(x)]_+\|_1$

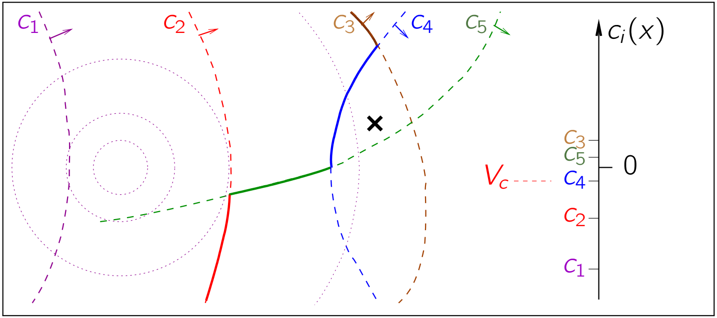
Critical scenario selections



$$\mathfrak{S}(x) = \{\mathcal{S} \in \mathfrak{S} : c_i(x) \leq V_c(x) \text{ for } i \in \mathcal{S}\} \qquad V_c(x) = c_{i(M)}(x)$$

- Then $\langle\langle [c(x)]_+ \rangle\rangle_M = \|[c_{\mathcal{S}}(x)]_+\|_1$ for all $\mathcal{S} \in \mathfrak{S}(x)$.
- Here: $\mathfrak{S}(x) = \{(1, 2, 5)\}$

Critical scenario selections



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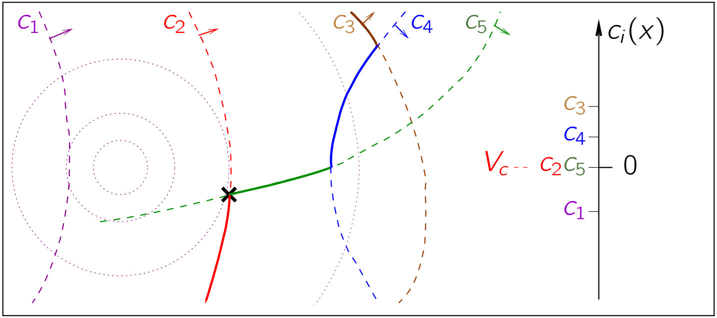
Critical scenario selections



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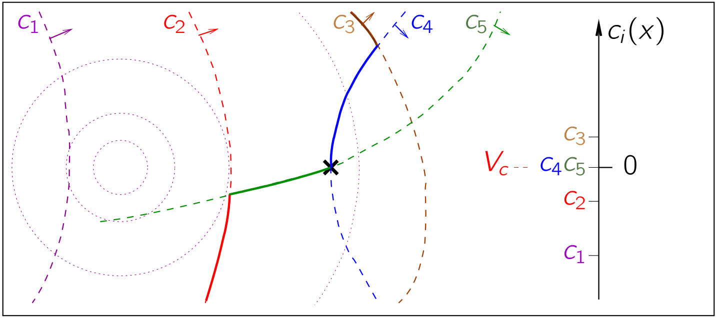

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- Then $\langle\langle [c(x)]_+ \rangle\rangle_M = \|[c_{\mathcal{S}}(x)]_+\|_1$ for all $\mathcal{S} \in \mathfrak{S}(x)$.
- Here: $\mathfrak{S}(x) = \{(1, 2, 4), (1, 2, 5)\}$

Exact penalty function for CCP

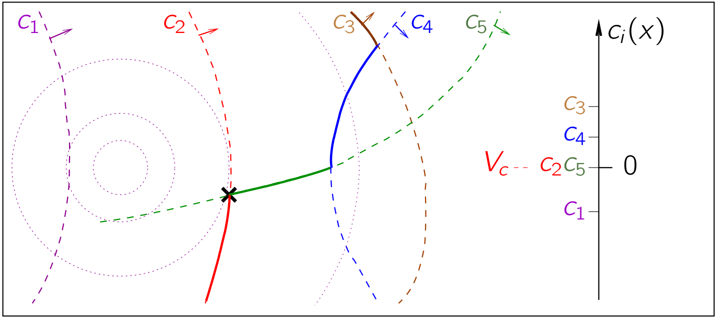
Penalty function for deterministic optimization with constraints in $\mathcal{S} \in \mathfrak{S}$:

$$\phi_{\mathcal{S}}(x) := f(x) + \rho \|[c_{\mathcal{S}}(x)]_+\|_1$$

Penalty function for CCP:

$$\begin{aligned}
\phi(x) &= f(x) + \rho \, \langle\!\langle [c(x)]_+ \rangle\!\rangle_M \\
&= f(x) + \rho \min_{\mathcal{S} \in \mathfrak{S}} \; \|[c_{\mathcal{S}}(x)]_+\|_1 \\
&= \min_{\mathcal{S} \in \mathfrak{S}} \; f(x) + \rho \|[c_{\mathcal{S}}(x)]_+\|_1 \\
&= \min_{\mathcal{S} \in \mathfrak{S}} \; \phi_{\mathcal{S}}(x) \\
&= \phi_{\tilde{\mathcal{S}}}(x) \qquad\qquad (\text{for all } \tilde{\mathcal{S}} \in \mathfrak{S}(x))
\end{aligned}$$

Equivalence of minimizers

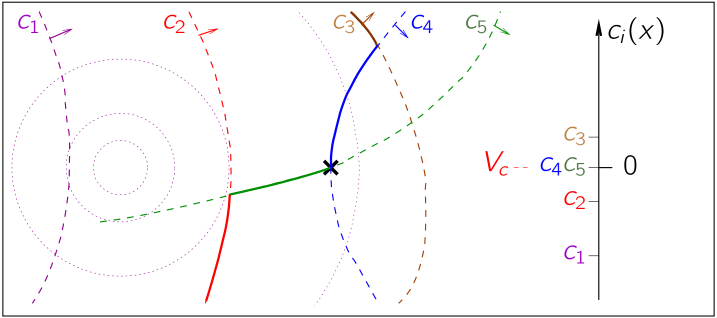


For $\rho > 0$ sufficiently large (and some CQ):

$$x_* \text{ local min of CCP}$$
$$\Updownarrow$$
$$\langle\langle [c(x_*)]_+ \rangle\rangle_M = 0 \text{ and } x_* \text{ local min of } \phi_{\mathcal{S}}(x) \text{ for all } \mathcal{S} \in \mathfrak{S}(x_*)$$

- Here $\mathfrak{S}(x_*) = \{(1, 2, 5)\}$ and x_* is local min of CCP

Equivalence of minimizers



For $\rho > 0$ sufficiently large (and some CQ):

$$x_* \text{ local min of CCP}$$
$$\Updownarrow$$
$$\langle\langle [c(x_*)]_+ \rangle\rangle_M = 0 \text{ and } x_* \text{ local min of } \phi_{\mathcal{S}}(x) \text{ for all } \mathcal{S} \in \mathfrak{S}(x_*)$$

- Here $\mathfrak{S}(x_*) = \{(1, 2, 4), (1, 2, 5)\}$ and x_* is NOT local min of CCP

Outline

Trust-region method for deterministic optimization (Sℓ_1QP)

$$\phi(x) = f(x) + \rho\|[c(x)]_+\|_1 \qquad (g_k = \nabla f(x_k), J_k = \nabla c(x_k)^T)$$

$$m_k(d) = f_k + g_k^T d + \tfrac{1}{2} d^T H_k d + \rho\|[c_k + J_k d]_+\|_1$$

At iterate x_k:

1. Compute step d_k from trust region subproblem (QP):

$$\boxed{\min_{d \in \mathbb{R}^n} \; m_k(d) \quad \text{s.t.} \quad \|d\|_\infty \le \delta_k}$$

2. Define ratio

$$r_k = \frac{\phi(x_k) - \phi(x_k + d_k)}{m_k(0) - m_k(d_k)}$$

3. Update iterate and trust region radius $(\mu \in (0,1))$
   - If $r_k \ge \mu$: $x_{k+1} \leftarrow x_k + d_k$ and $\delta_{k+1} \leftarrow 2\delta_k$
   - If $r_k < \mu$: $x_{k+1} \leftarrow x_k$ and $\delta_{k+1} \leftarrow 0.5\|d_k\|_\infty$

Trust-region method for CCP

$$\phi(x) = f(x) + \rho\langle\langle [c(x)]_+ \rangle\rangle_M$$
$$m_k(d) = f_k + g_k^T d + \tfrac{1}{2} d^T H_k d + \rho\langle\langle [c_k + J_k d]_+ \rangle\rangle_M$$

At iterate x_k:

1. Compute step d_k from trust region subproblem (QCCP):

$$\min_{d\in\mathbb{R}^n} \ m_k(d) \quad \text{s.t.} \quad \|d\|_\infty \le \delta_k$$

2. Define ratio

$$r_k = \frac{\phi(x_k) - \phi(x_k + d_k)}{m_k(0) - m_k(d_k)}$$

3. Update iterate and trust region radius $(\mu \in (0,1))$
   - If $r_k \ge \mu$: $x_{k+1} \leftarrow x_k + d_k$ and $\delta_{k+1} \leftarrow \max\{2\delta_k, \delta_{\text{reset}}\}$ $(\delta_{\text{reset}} > 0)$
   - If $r_k < \mu$: $x_{k+1} \leftarrow x_k$ and $\delta_{k+1} = 0.5\|d_k\|_\infty$

Solving the trust region subproblem

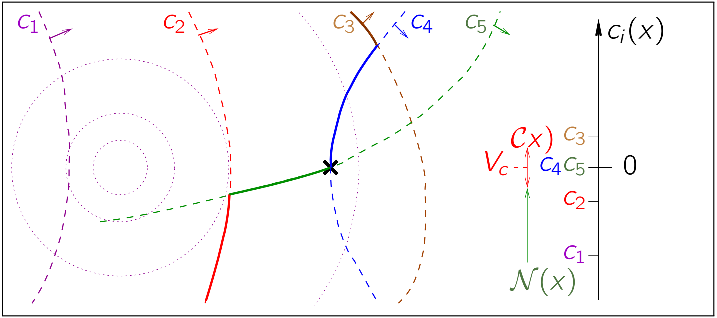
$$m_k(d) = f_k + g_k^T d + \tfrac{1}{2} d^T H_k d + \rho \, \langle\!\langle [c_k + J_k d]_+ \rangle\!\rangle_M$$
$$= f_k + g_k^T d + \tfrac{1}{2} d^T H_k d + \rho \min_{\mathcal{S} \in \mathfrak{S}} \| [c_{\mathcal{S},k} + J_{\mathcal{S},k} d]_+ \|_1$$

MIQP formulation:

$$\min_{d,s,z} \; f_k + g_k^T d + \tfrac{1}{2} d^T H_k d + \rho \sum_{i=1}^{N} s_i$$
$$\text{s.t. } c_i(x_k) + \nabla c_i(x_k)^T d \le s_i + \text{bigM}(1 - z_i)$$
$$\sum_{i=1}^{N} z_i = M, \quad s \ge 0, \quad \|d\|_\infty \le \delta_k, \quad z \in \{0,1\}^N$$

- big-M formulation is computationally expensive
- Better alternative: Use B&C method of [Luedtke 14]
- Can we reduce the number of scenarios to consider?

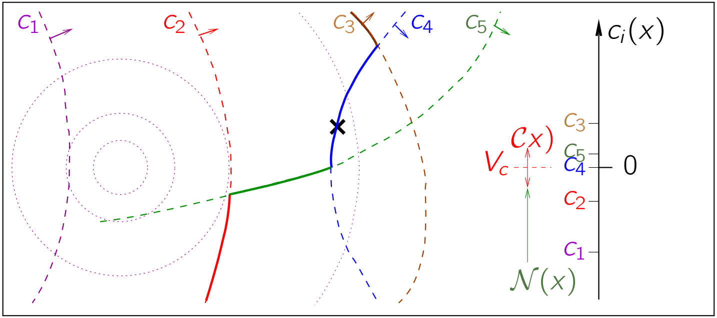
ϵ-critical scenario selections



$$\mathcal{C}(x,\epsilon) = \{i \in \{1,\ldots,N\} : |c_i(x) - V_c(x)| \le \epsilon\}$$
$$\mathcal{N}(x,\epsilon) = \{i \in \{1,\ldots,N\} : c_i(x) \le V_c(x) - \epsilon\}$$
$$\mathfrak{S}(x,\epsilon) = \{\mathcal{S} \in \mathfrak{S} : \mathcal{N}(x,\epsilon) \subseteq \mathcal{S} \text{ and } \mathcal{S} \subseteq \mathcal{N}(x,\epsilon) \cup \mathcal{C}(x,\epsilon)\}$$

- Here: $\mathfrak{S}(x_*,\epsilon) = \{(1,2,4),(1,2,5)\}$

ϵ-critical scenario selections



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- Here: $\mathfrak{S}(x_k,\epsilon) = \{(1,2,4),(1,2,5)\}$

Subproblem with ϵ-critical scenario selections

$$\min_{d \in \mathbb{R}^n} \; f_k + g_k^T d + \tfrac{1}{2} d^T H_k d + \rho \min_{\mathcal{S} \in \mathfrak{S}(x_k, \epsilon)} \| [c_{\mathcal{S},k} + J_{\mathcal{S},k} d]_+ \|_1$$

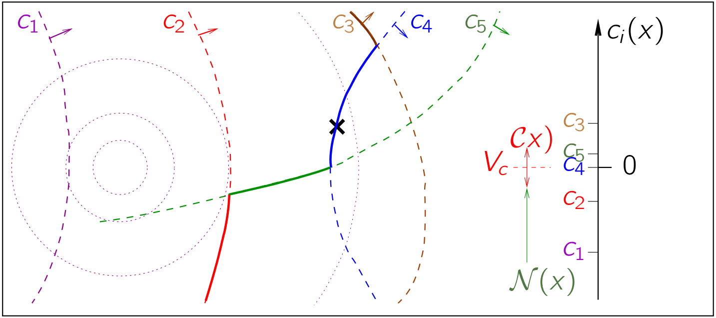
$$\text{s.t. } \|d\|_\infty \le \delta_k$$

- Computational effort can be tuned with ϵ
  - Small ϵ: Easier subproblems
  - Large ϵ: "Broader view" of feasible region

Theorem 1

Suppose $f \in \mathcal{C}^1$, $c(\cdot, \xi) \in \mathcal{C}^1$, $\{H_k\}$ bounded, and $\phi(x_k)$ bounded below. Then, every limit point x_* of $\{x_k\}$ is a stationary point of $\phi_{\mathcal{S}}$ for all $\mathcal{S} \in \mathfrak{S}(x_*)$.

Joint chance constraints



- Joint chance constraints: $m > 1$
- Combined constraint satisfaction:

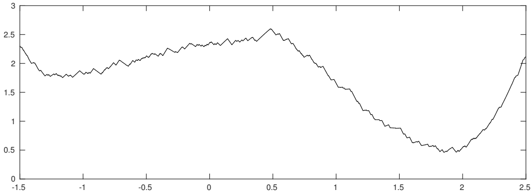
$$V_i(x) = \begin{cases} \|[c_i(x)]_+\|_1 & \text{if } c_i(x) \not\leq 0 \\ \max\{c_{ij}(x) : j = 1, \ldots, m\} & \text{if } c_i(x) \leq 0 \end{cases}$$

Outline

Numerical experiments

- Matlab implementation with AMPL interface (suffixes)
- MIQPs solved by CPLEX
  - take incumbent after 5 min time limit (parallel with 8 threads)
- H_k: Hessian of Lagrangian for current scenario selection
- $\gamma \in (0, 1]$: include $\lceil \gamma N \rceil$ scenarios below and above $V_c(x_k)$ in $\mathcal{C}$

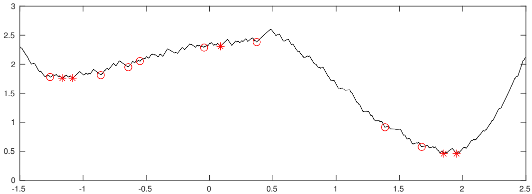
"Poor" local minimizers



$$\min_{x,z} \; z$$
$$\text{s.t. } \mathbb{P}[0.25x^4 - \tfrac{1}{3}x^3 - x^2 + 0.2x - 19.5 + \xi_1 x + \xi_1 \xi_0 \leq z] \geq 0.95$$

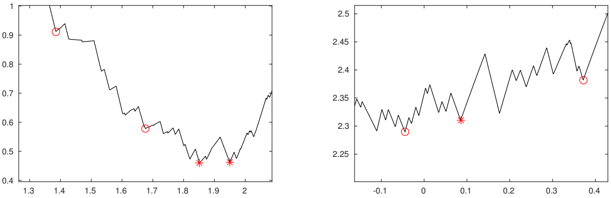
- ξ_0 from $U[-12, 12]$ and ξ_1 from $U[-3, 3]$
- $N = 5{,}000$

“Poor” local minimizers



γ		0.001	0.002	0.005	0.010	0.050	0.100	0.200	1.000
x_0	N	11	21	51	101	501	751	1251	5000
−1.5		*1.776*	*1.776*	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**
−1.0		**1.754**	**1.760**	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**
−0.5		2.049	1.951	1.814	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**
0.0		2.289	2.289	1.814	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**
0.5		2.382	2.382	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**
1.0		0.912	**0.460**	**0.460**	**0.460**	**0.463**	**0.463**	**0.463**	**0.463**
1.5		0.579	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**
2.0		**0.463**	**0.463**	**0.460**	**0.460**	**0.463**	**0.463**	**0.463**	**0.463**
x^{bad}		2.310	2.289	1.814	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**

“Poor” local minimizers



x_0 \ γ / N	0.001 / 11	0.002 / 21	0.005 / 51	0.010 / 101	0.050 / 501	0.100 / 751	0.200 / 1251	1.000 / 5000
-1.5	*1.776*	*1.776*	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**
-1.0	**1.754**	**1.760**	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**
-0.5	2.049	1.951	1.814	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**
0.0	2.289	2.289	1.814	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**
0.5	2.382	2.382	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**
1.0	0.912	**0.460**	**0.460**	**0.460**	**0.463**	**0.463**	**0.463**	**0.463**
1.5	0.579	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**	**0.460**
2.0	**0.463**	**0.463**	**0.460**	**0.460**	**0.463**	**0.463**	**0.463**	**0.463**
x^{bad}	2.310	2.289	1.814	**1.754**	**1.754**	**1.754**	**1.754**	**1.754**

Cash flow problem [Dentcheva et al. 03]

- Invest capital over time horizon into different investment options
- Make sure we can pay random liabilities (with 95% prob.)
- Maximize final cash
- Interest rates depend on amount of investment
  - This makes the problem nonconvex
- 5 random instances per size

Results

N	γ	instances solved	iter	$\|\mathcal{C}_k\|$	changes in $\mathcal{C}_k$	% improve over robust	time in secs
500	0.001	5	9.84	9.40	6.14	5.7223 (5)	25.95
500	0.005	5	8.19	11.84	8.68	5.9602 (5)	37.45
500	0.050	5	7.04	51.00	13.06	6.0349 (5)	52.96
500	1.000	5	7.04	500.00	15.52	6.0349 (5)	260.08
1000	0.001	5	15.98	10.90	15.15	7.4040 (5)	117.13
1000	0.005	5	12.29	15.22	23.72	7.7296 (5)	147.30
1000	0.050	5	8.12	101.00	34.29	7.9134 (5)	313.17
1000	1.000	5	7.43	1000.00	57.14	7.9134 (5)	819.24
2000	0.001	5	21.94	13.84	43.67	9.7347 (4)	652.19
2000	0.005	4	12.85	23.01	56.61	9.9029 (4)	513.20
2000	0.050	4	9.90	201.00	88.99	10.1009 (4)	1099.41
2000	1.000	5	14.33	2000.00	129.18	10.0924 (4)	3519.53

- 96 vars, 20 chance constr, 150 non-chance constr
- Size for $N = 2000$ and $\gamma = 0.20$:
  - MIQP has about 40,000 constraints and 500 binary variables

Outline

Conclusions

- SQP-type trust-region algorithm
  - Exact penalty function for chance-constrained NLP
- Computational effort of subproblem can be tuned
  - Small ϵ: Reduce solution time
  - Large ϵ: Escape spurious local solutions
- Potential improvements
  - Use efficient branch-and-cut subproblem (e.g., [Luedtke 12], ...)
  - Adaptive choices of ρ and/or ϵ or γ
  - Adaptive sample size
    - At beginning, consider small subset of scenarios and choose large ϵ
    - Refine SAA discretization with proximity to solution
  - Heuristic solutions of subproblem
    - Avoid solution of mixed-integer problem
    - E.g., progress along successively improving scenario selections
  - Ignore "very satisfied" constraints in subproblem
    - Do not include scenarios with $c_i(x) \ll 0$