Inexact Newton Methods and Nonlinear Constrained Optimization

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Outline

Outline

PDE-Constrained Optimization

Newton's method

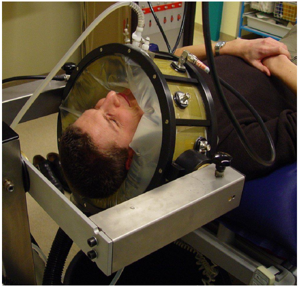
Inexactness

Experimental results

Conclusion and final remarks

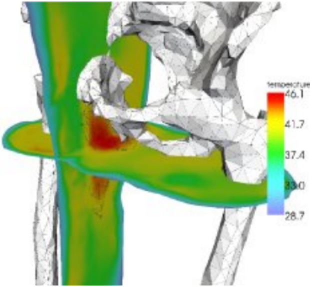
Hyperthermia treatment

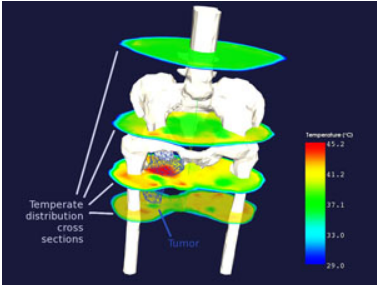
- Regional hyperthermia is a cancer therapy that aims at heating large and deeply seated tumors by means of radio wave adsorption
- Results in the killing of tumor cells and makes them more susceptible to other accompanying therapies; e.g., chemotherapy



Hyperthermia treatment planning

- Computer modeling can be used to help plan the therapy for each patient, and it opens the door for numerical optimization
- The goal is to heat the tumor to a target temperature of 43°C while minimizing damage to nearby cells





PDE-constrained optimization

$$\begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & c_{\mathcal{E}}(x) = 0 \\ & c_{\mathcal{I}}(x) \geq 0 \end{aligned}$$

- Problem is infinite-dimensional
- Controls and states: $x = (u, y)$
- Solution methods integrate
  - numerical simulation
  - problem structure
  - optimization algorithms

Algorithmic frameworks

We hear the phrases:

- Discretize-then-optimize
- Optimize-then-discretize

I prefer:

- Discretize the optimization problem

$$\boxed{\begin{array}{l}\min\ f(x)\\ \text{s.t.}\ c(x)=0\end{array}} \Rightarrow \boxed{\begin{array}{l}\min\ f_h(x)\\ \text{s.t.}\ c_h(x)=0\end{array}}$$

- Discretize the optimality conditions

$$\boxed{\begin{array}{l}\min\ f(x)\\ \text{s.t.}\ c(x)=0\end{array}} \Rightarrow \boxed{\begin{bmatrix}\nabla f+\langle A,\lambda\rangle\\ c\end{bmatrix}=0} \Rightarrow \boxed{\begin{bmatrix}(\nabla f+\langle A,\lambda\rangle)_h\\ c_h\end{bmatrix}=0}$$

- Discretize the search direction computation

Algorithms

- Nonlinear elimination

$$\boxed{\begin{aligned}&\min_{u,y} f(u,y)\\ &\text{s.t. } c(u,y)=0\end{aligned}} \Rightarrow \boxed{\min_{u} f(u,y(u))} \Rightarrow \boxed{\nabla_u f + \nabla_u y^T \nabla_y f = 0}$$

- Reduced-space methods

$$\begin{aligned} d_y &: \text{ toward satisfying the constraints}\\ \lambda &: \text{ Lagrange multiplier estimates}\\ d_u &: \text{ toward optimality}\end{aligned}$$

- Full-space methods

$$\begin{bmatrix} H_u & 0 & A_u^T \\ 0 & H_y & A_y^T \\ A_u & A_y & 0 \end{bmatrix} \begin{bmatrix} d_u \\ d_y \\ \delta \end{bmatrix} = - \begin{bmatrix} \nabla_u f + A_u^T \lambda \\ \nabla_y f + A_y^T \lambda \\ c \end{bmatrix}$$

Outline

PDE-Constrained Optimization

Newton's method

Inexactness

Experimental results

Conclusion and final remarks

Nonlinear equations

- Newton's method

$$\boxed{\mathcal{F}(x) = 0} \Rightarrow \boxed{\nabla \mathcal{F}(x_k) d_k = -\mathcal{F}(x_k)}$$

- Judge progress by the merit function

$$\phi(x) \triangleq \frac{1}{2}\|\mathcal{F}(x)\|^2$$

- Direction is one of descent since

$$\nabla \phi(x_k)^T d_k = \mathcal{F}(x_k)^T \nabla \mathcal{F}(x_k) d_k = -\|\mathcal{F}(x_k)\|^2 < 0$$

(Note the consistency between the step computation and merit function!)

Equality constrained optimization

- Consider

$$\min_{x \in \mathbb{R}^n} f(x)$$
$$\text{s.t. } c(x) = 0$$

- Lagrangian is

$$\mathcal{L}(x, \lambda) \triangleq f(x) + \lambda^T c(x)$$

so the first-order optimality conditions are

$$\nabla \mathcal{L}(x, \lambda) = \begin{bmatrix} \nabla f(x) + \nabla c(x)\lambda \\ c(x) \end{bmatrix} \triangleq \mathcal{F}(x, \lambda) = 0$$

Merit function

- Simply minimizing

$$\varphi(x,\lambda) = \tfrac{1}{2}\|\mathcal{F}(x,\lambda)\|^2 = \tfrac{1}{2}\left\| \begin{bmatrix} \nabla f(x) + \nabla c(x)\lambda \\ c(x) \end{bmatrix} \right\|^2$$

  is generally inappropriate for constrained optimization

- We use the merit function

$$\phi(x;\pi) \triangleq f(x) + \pi\|c(x)\|$$

  where π is a penalty parameter

Minimizing a penalty function

Consider the penalty function for

$$\min\ (x-1)^2, \text{ s.t. } x = 0 \quad \text{i.e.} \quad \phi(x;\pi) = (x-1)^2 + \pi|x|$$

for different values of the penalty parameter π

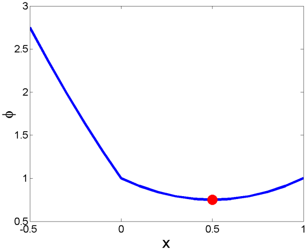


Figure: $\pi = 1$

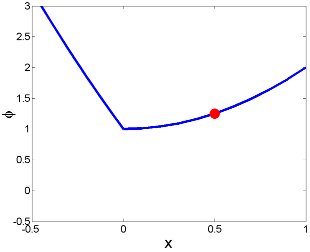


Figure: $\pi = 2$

Algorithm 0: Newton method for optimization

(Assume the problem is sufficiently convex and regular)
for $k = 0, 1, 2, \ldots$

- Solve the primal-dual (Newton) equations

$$\begin{bmatrix} H(x_k, \lambda_k) & \nabla c(x_k) \\ \nabla c(x_k)^T & 0 \end{bmatrix} \begin{bmatrix} d_k \\ \delta_k \end{bmatrix} = - \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k)\lambda_k \\ c(x_k) \end{bmatrix}$$

- Increase π, if necessary, so that $D\phi_k(d_k; \pi_k) \ll 0$ (e.g., $\pi_k \geq \|\lambda_k + \delta_k\|$)
- Backtrack from $\alpha_k \leftarrow 1$ to satisfy the Armijo condition

$$\phi(x_k + \alpha_k d_k; \pi_k) \leq \phi(x_k; \pi_k) + \eta \alpha_k D\phi_k(d_k; \pi_k)$$

- Update iterate $(x_{k+1}, \lambda_{k+1}) \leftarrow (x_k, \lambda_k) + \alpha_k(d_k, \delta_k)$

Convergence of Algorithm 0

Assumption

The sequence $\{(x_k, \lambda_k)\}$ is contained in a convex set Ω over which f, c, and their first derivatives are bounded and Lipschitz continuous. Also,

- *(Regularity) $\nabla c(x_k)^T$ has full row rank with singular values bounded below by a positive constant*
- *(Convexity) $u^T H(x_k, \lambda_k) u \geq \mu \|u\|^2$ for $\mu > 0$ for all $u \in \mathbb{R}^n$ satisfying $u \neq 0$ and $\nabla c(x_k)^T u = 0$*

Theorem

(Han (1977)) The sequence $\{(x_k, \lambda_k)\}$ yields the limit

$$\lim_{k \to \infty} \left\| \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k)\lambda_k \\ c(x_k) \end{bmatrix} \right\| = 0$$

Outline

Large-scale primal-dual algorithms

- Computational issues:
  - Large matrices to be stored
  - Large matrices to be factored
- Algorithmic issues:
  - The problem may be nonconvex
  - The problem may be ill-conditioned
- Computational/Algorithmic issues:
  - No matrix factorizations makes difficulties more difficult

Nonlinear equations

- Compute

$$\nabla \mathcal{F}(x_k) d_k = -\mathcal{F}(x_k) + r_k$$

requiring (Dembo, Eisenstat, Steihaug (1982))

$$\|r_k\| \leq \kappa \|\mathcal{F}(x_k)\|, \quad \kappa \in (0,1)$$

- Progress judged by the merit function

$$\phi(x) \triangleq \tfrac{1}{2}\|\mathcal{F}(x)\|^2$$

- Again, note the consistency...

$$\nabla \phi(x_k)^T d_k = \mathcal{F}(x_k)^T \nabla \mathcal{F}(x_k) d_k = -\|\mathcal{F}(x_k)\|^2 + \mathcal{F}(x_k)^T r_k \leq (\kappa - 1)\|\mathcal{F}(x_k)\|^2 < 0$$

Optimization

- Compute

$$\begin{bmatrix} H(x_k,\lambda_k) & \nabla c(x_k) \\ \nabla c(x_k)^T & 0 \end{bmatrix} \begin{bmatrix} d_k \\ \delta_k \end{bmatrix} = - \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k)\lambda_k \\ c(x_k) \end{bmatrix} + \begin{bmatrix} \rho_k \\ r_k \end{bmatrix}$$

satisfying

$$\left\| \begin{bmatrix} \rho_k \\ r_k \end{bmatrix} \right\| \leq \kappa \left\| \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k)\lambda_k \\ c(x_k) \end{bmatrix} \right\|, \quad \kappa \in (0,1)$$

- If κ is not sufficiently small (e.g., 10^{-3} vs. 10^{-12}), then d_k may be an ascent direction for our merit function; i.e.,

$$D\phi_k(d_k;\pi_k) > 0 \quad \text{for all } \pi_k \geq \pi_{k-1}$$

- Our work begins here... inexact Newton methods for optimization
- We cover the convex case, nonconvexity, irregularity, inequality constraints

Model reductions

- Define the model of $\phi(x;\pi)$:

$$m(d;\pi) \triangleq f(x) + \nabla f(x)^T d + \pi(\|c(x) + \nabla c(x)^T d\|)$$

- d_k is acceptable if

$$\begin{aligned}\Delta m(d_k;\pi_k) &\triangleq m(0;\pi_k) - m(d_k;\pi_k) \\ &= -\nabla f(x_k)^T d_k + \pi_k(\|c(x_k)\| - \|c(x_k) + \nabla c(x_k)^T d_k\|) \gg 0\end{aligned}$$

- This ensures $D\phi_k(d_k;\pi_k) \ll 0$ (and more)

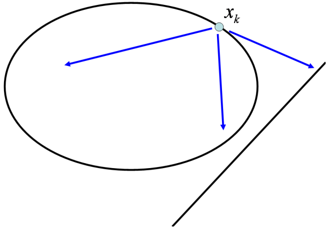
Termination test 1

The search direction (d_k, δ_k) is acceptable if

$$\left\| \begin{bmatrix} \rho_k \\ r_k \end{bmatrix} \right\| \leq \kappa \left\| \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k)\lambda_k \\ c(x_k) \end{bmatrix} \right\|, \quad \kappa \in (0,1)$$

and if for $\pi_k = \pi_{k-1}$ and some $\sigma \in (0,1)$ we have

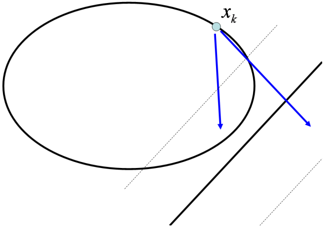
$$\Delta m(d_k;\pi_k) \geq \underbrace{\max\{\tfrac{1}{2}d_k^T H(x_k,\lambda_k)d_k, 0\} + \sigma\pi_k \max\{\|c(x_k)\|, \|r_k\| - \|c(x_k)\|\}}_{\geq 0 \text{ for any } d}$$



Termination test 2

The search direction (d_k, δ_k) is acceptable if

$$\|\rho_k\| \leq \beta\|c(x_k)\|, \quad \beta > 0$$
$$\text{and} \quad \|r_k\| \leq \epsilon\|c(x_k)\|, \quad \epsilon \in (0,1)$$



Increasing the penalty parameter π then yields

$$\Delta m(d_k;\pi_k) \geq \underbrace{\max\{\tfrac{1}{2}d_k^T H(x_k,\lambda_k)d_k, 0\} + \sigma\pi_k\|c(x_k)\|}_{\geq 0 \text{ for any } d}$$

Algorithm 1: Inexact Newton for optimization

(Byrd, Curtis, Nocedal (2008))

for $k = 0, 1, 2, \ldots$

- Iteratively solve

$$\begin{bmatrix} H(x_k, \lambda_k) & \nabla c(x_k) \\ \nabla c(x_k)^T & 0 \end{bmatrix} \begin{bmatrix} d_k \\ \delta_k \end{bmatrix} = - \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k)\lambda_k \\ c(x_k) \end{bmatrix}$$

  until termination test 1 or 2 is satisfied

- If only termination test 2 is satisfied, increase π so

$$\pi_k \geq \max\left\{ \pi_{k-1}, \frac{\nabla f(x_k)^T d_k + \max\{\frac{1}{2} d_k^T H(x_k, \lambda_k) d_k, 0\}}{(1-\tau)(\|c(x_k)\| - \|r_k\|)} \right\}$$

- Backtrack from $\alpha_k \leftarrow 1$ to satisfy

$$\phi(x_k + \alpha_k d_k; \pi_k) \leq \phi(x_k; \pi_k) - \eta \alpha_k \Delta m(d_k; \pi_k)$$

- Update iterate $(x_{k+1}, \lambda_{k+1}) \leftarrow (x_k, \lambda_k) + \alpha_k (d_k, \delta_k)$

Convergence of Algorithm 1

Assumption

The sequence $\{(x_k, \lambda_k)\}$ is contained in a convex set Ω over which f, c, and their first derivatives are bounded and Lipschitz continuous. Also,

- *(Regularity) $\nabla c(x_k)^T$ has full row rank with singular values bounded below by a positive constant*
- *(Convexity) $u^T H(x_k, \lambda_k) u \geq \mu \|u\|^2$ for $\mu > 0$ for all $u \in \mathbb{R}^n$ satisfying $u \neq 0$ and $\nabla c(x_k)^T u = 0$*

Theorem

(Byrd, Curtis, Nocedal (2008)) The sequence $\{(x_k, \lambda_k)\}$ yields the limit

$$\lim_{k \to \infty} \left\| \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k) \lambda_k \\ c(x_k) \end{bmatrix} \right\| = 0$$

Handling nonconvexity and rank deficiency

- There are two assumptions we aim to drop:
  - *(Regularity) $\nabla c(x_k)^T$ has full row rank with singular values bounded below by a positive constant*
  - *(Convexity) $u^T H(x_k, \lambda_k) u \geq \mu \|u\|^2$ for $\mu > 0$ for all $u \in \mathbb{R}^n$ satisfying $u \neq 0$ and $\nabla c(x_k)^T u = 0$*

  e.g., the problem is not regular if it is infeasible, and it is not convex if there are maximizers and/or saddle points
- Without them, Algorithm 1 may stall or may not be well-defined

No factorizations means no clue

- We might not store or factor

$$\begin{bmatrix} H(x_k, \lambda_k) & \nabla c(x_k) \\ \nabla c(x_k)^T & 0 \end{bmatrix}$$

so we might not know if the problem is nonconvex or ill-conditioned

- Common practice is to perturb the matrix to be

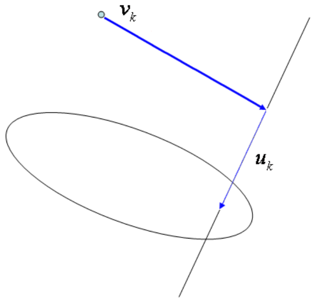
$$\begin{bmatrix} H(x_k, \lambda_k) + \xi_1 I & \nabla c(x_k) \\ \nabla c(x_k)^T & -\xi_2 I \end{bmatrix}$$

where ξ_1 convexifies the model and ξ_2 regularizes the constraints

- Poor choices of ξ_1 and ξ_2 can have terrible consequences in the algorithm

Our approach for global convergence

- Decompose the direction d_k into a normal component (toward the constraints) and a tangential component (toward optimality)



- We impose a specific type of trust region constraint on the v_k step in case the constraint Jacobian is (near) rank deficient

Handling nonconvexity

- In computation of $d_k = v_k + u_k$, convexify the Hessian as in

$$\begin{bmatrix} H(x_k, \lambda_k) + \xi_1 I & \nabla c(x_k) \\ \nabla c(x_k)^T & 0 \end{bmatrix}$$

by monitoring iterates

- Hessian modification strategy: Increase ξ_1 whenever

$$\begin{aligned} \|u_k\|^2 &> \psi \|v_k\|^2, \quad \psi > 0 \\ \tfrac{1}{2} u_k^T (H(x_k, \lambda_k) + \xi_1 I) u_k &< \theta \|u_k\|^2, \quad \theta > 0 \end{aligned}$$

Algorithm 2: Inexact Newton (Regularized)

(Curtis, Nocedal, Wächter (2009))

for $k = 0, 1, 2, \ldots$

- Approximately solve

$$\min \ \tfrac{1}{2}\|c(x_k) + \nabla c(x_k)^T v\|^2, \quad \text{s.t. } \|v\| \le \omega \|(\nabla c(x_k))c(x_k)\|$$

  to compute v_k satisfying Cauchy decrease

- Iteratively solve

$$\begin{bmatrix} H(x_k, \lambda_k) + \xi_1 I & \nabla c(x_k) \\ \nabla c(x_k)^T & 0 \end{bmatrix} \begin{bmatrix} d_k \\ \delta_k \end{bmatrix} = - \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k)\lambda_k \\ -\nabla c(x_k)^T v_k \end{bmatrix}$$

  until termination test 1 or 2 is satisfied, increasing ξ_1 as described

- If only termination test 2 is satisfied, increase π so

$$\pi_k \ge \max \left\{ \pi_{k-1}, \frac{\nabla f(x_k)^T d_k + \max\{\frac{1}{2} u_k^T (H(x_k, \lambda_k) + \xi_1 I) u_k, \theta \|u_k\|^2\}}{(1-\tau)(\|c(x_k)\| - \|c(x_k) + \nabla c(x_k)^T d_k\|)} \right\}$$

- Backtrack from $\alpha_k \leftarrow 1$ to satisfy

$$\phi(x_k + \alpha_k d_k; \pi_k) \le \phi(x_k; \pi_k) - \eta \alpha_k \Delta m(d_k; \pi_k)$$

- Update iterate $(x_{k+1}, \lambda_{k+1}) \leftarrow (x_k, \lambda_k) + \alpha_k (d_k, \delta_k)$

Convergence of Algorithm 2

Assumption

The sequence $\{(x_k, \lambda_k)\}$ is contained in a convex set Ω over which f, c, and their first derivatives are bounded and Lipschitz continuous

Theorem

(Curtis, Nocedal, Wächter (2009)) If all limit points of $\{\nabla c(x_k)^T\}$ have full row rank, then the sequence $\{(x_k, \lambda_k)\}$ yields the limit

$$\lim_{k\to\infty} \left\| \begin{bmatrix} \nabla f(x_k) + \nabla c(x_k)\lambda_k \\ c(x_k) \end{bmatrix} \right\| = 0.$$

Otherwise,

$$\lim_{k\to\infty} \|(\nabla c(x_k))c(x_k)\| = 0$$

and if $\{\pi_k\}$ is bounded, then

$$\lim_{k\to\infty} \|\nabla f(x_k) + \nabla c(x_k)\lambda_k\| = 0$$

Handling inequalities

- Interior point methods are attractive for large applications
- Line-search interior point methods that enforce

$$c(x_k) + \nabla c(x_k)^T d_k = 0$$

  may fail to converge globally (Wächter, Biegler (2000))
- Fortunately, the trust region subproblem we use to regularize the constraints also saves us from this type of failure!

Algorithm 2 (Interior-point version)

- Apply Algorithm 2 to the logarithmic-barrier subproblem

$$\min \ f(x) - \mu \sum_{i=1}^{q} \ln s^i, \quad \text{s.t. } c_{\mathcal{E}}(x) = 0, \ c_{\mathcal{I}}(x) - s = 0$$

  for $\mu \to 0$

- Define

$$\begin{bmatrix} H(x_k, \lambda_{\mathcal{E},k}, \lambda_{\mathcal{I},k}) & 0 & \nabla c_{\mathcal{E}}(x_k) & \nabla c_{\mathcal{I}}(x_k) \\ 0 & \mu I & 0 & -S_k \\ \nabla c_{\mathcal{E}}(x_k)^T & 0 & 0 & 0 \\ \nabla c_{\mathcal{I}}(x_k)^T & -S_k & 0 & 0 \end{bmatrix} \begin{bmatrix} d_k^x \\ d_k^s \\ \delta_{\mathcal{E},k} \\ \delta_{\mathcal{I},k} \end{bmatrix}$$

  so that the iterate update has

$$\begin{bmatrix} x_{k+1} \\ s_{k+1} \end{bmatrix} \leftarrow \begin{bmatrix} x_k \\ s_k \end{bmatrix} + \alpha_k \begin{bmatrix} d_k^x \\ S_k d_k^s \end{bmatrix}$$

- Incorporate a fraction-to-the-boundary rule in the line search and a slack reset in the algorithm to maintain $s \geq \max\{0, c_{\mathcal{I}}(x)\}$

Convergence of Algorithm 2 (Interior-point)

Assumption

The sequence $\{(x_k, \lambda_{\mathcal{E},k}, \lambda_{\mathcal{I},k})\}$ is contained in a convex set Ω over which f, $c_{\mathcal{E}}$, $c_{\mathcal{I}}$, and their first derivatives are bounded and Lipschitz continuous

Theorem

(Curtis, Schenk, Wächter (2009))

- *For a given μ, Algorithm 2 yields the same limits as in the equality constrained case*
- *If Algorithm 2 yields a sufficiently accurate solution to the barrier subproblem for each $\{\mu_j\} \to 0$ and if the linear independence constraint qualification (LICQ) holds at a limit point $\bar{x}$ of $\{x_j\}$, then there exist Lagrange multipliers $\bar{\lambda}$ such that the first-order optimality conditions of the nonlinear program are satisfied*

Outline

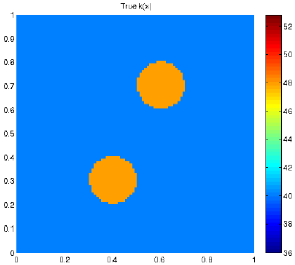
Implementation details

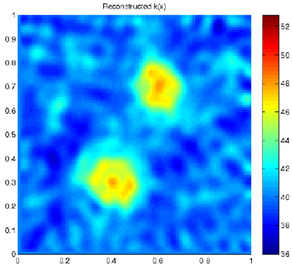
- Incorporated in `IPOPT` software package (Wächter)
  - `inexact_algorithm yes`
- Linear systems solved with `PARDISO` (Schenk)
  - SQMR (Freund (1994))
- Preconditioning in `PARDISO`
  - incomplete multilevel factorization with inverse-based pivoting
  - stabilized by symmetric-weighted matchings
- Optimality tolerance: 1e-8

CUTEr and COPS collections

- 745 problems written in AMPL
- 645 solved successfully
- 42 "real" failures
- Robustness between 87%-94%
- Original `IPOPT`: 93%

Helmholtz





N	n	p	q	# iter	CPU sec (per iter)
32	14724	13824	1800	37	807.823 (21.833)
64	56860	53016	7688	25	3741.42 (149.66)
128	227940	212064	31752	20	54581.8 (2729.1)

Boundary control

$$
\begin{aligned}
\min\ & \tfrac{1}{2}\int_\Omega (y(x) - y_t(x))^2 dx \\
\text{s.t.}\ & -\nabla \cdot (e^{y(x)} \cdot \nabla y(x)) = 20 \quad \text{in } \Omega \\
& y(x) = u(x) \quad \text{on } \partial\Omega \\
& 2.5 \le u(x) \le 3.5 \quad \text{on } \partial\Omega
\end{aligned}
$$

where

$$y_t(x) = 3 + 10x_1(x_1 - 1)x_2(x_2 - 1)\sin(2\pi x_3)$$

N	n	p	q	# iter	CPU sec (per iter)
16	4096	2744	2704	13	2.8144 (0.2165)
32	32768	27000	11536	13	103.65 (7.9731)
64	262144	238328	47632	14	5332.3 (380.88)

Original IPOPT with $N = 32$ requires 238 seconds per iteration

Hyperthermia Treatment Planning

$$\begin{aligned} &\min\ \tfrac{1}{2}\int_\Omega (y(x)-y_t(x))^2 dx \\ &\text{s.t.}\ -\Delta y(x) - 10(y(x)-37) = u^* M(x) u \ \text{ in } \Omega \\ &\qquad 37.0 \le y(x) \le 37.5 \ \text{ on } \partial\Omega \\ &\qquad 42.0 \le y(x) \le 44.0 \ \text{ in } \Omega_0 \end{aligned}$$

where

$$u_j = a_j e^{i\phi_j}, \quad M_{jk}(x) = < E_j(x), E_k(x) >, \quad E_j = \sin(j x_1 x_2 x_3 \pi)$$

N	n	p	q	# iter	CPU sec (per iter)
16	4116	2744	2994	68	22.893 (0.3367)
32	32788	27000	13034	51	3055.9 (59.920)

Original `IPOPT` with $N = 32$ requires 408 seconds per iteration

Groundwater modeling

$$\begin{aligned}
\min\ & \tfrac{1}{2}\textstyle\int_\Omega (y(x) - y_t(x))^2 dx + \tfrac{1}{2}\alpha \textstyle\int_\Omega [\beta(u(x) - u_t(x))^2 + |\nabla(u(x) - u_t(x))|^2] dx \\
\text{s.t. } & -\nabla \cdot (e^{u(x)} \cdot \nabla y_i(x)) = q_i(x) \ \text{ in } \Omega, \quad i = 1, \ldots, 6 \\
& \nabla y_i(x) \cdot n = 0 \ \text{ on } \partial\Omega \\
& \int_\Omega y_i(x) dx = 0, \quad i = 1, \ldots, 6 \\
& -1 \leq u(x) \leq 2 \ \text{ in } \Omega
\end{aligned}$$

where

$$q_i = 100 \sin(2\pi x_1) \sin(2\pi x_2) \sin(2\pi x_3)$$

N	n	p	q	# iter	CPU sec (per iter)
16	28672	24576	8192	18	206.416 (11.4676)
32	229376	196608	65536	20	1963.64 (98.1820)
64	1835008	1572864	524288	21	134418. (6400.85)

Original IPOPT with $N = 32$ requires approx. 20 hours for the first iteration

Outline

PDE-Constrained Optimization

Newton's method

Inexactness

Experimental results

Conclusion and final remarks

Conclusion and final remarks

- PDE-Constrained optimization is an active and exciting area
- Inexact Newton method with theoretical foundation
- Convergence guarantees are as good as exact methods, sometimes better
- Numerical experiments are promising so far, and more to come