A New Penalty-SQP Method

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Outline

Background and Motivation

A New Penalty-SQP Method

Illustration of Numerical Results

Final Remarks

Sequential Quadratic Programming

We solve the nonlinear optimization problem

$$\begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & b(x) \geq 0 \\ & c(x) = 0, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $b : \mathbb{R}^n \to \mathbb{R}^l$ and $c : \mathbb{R}^n \to \mathbb{R}^m$ are smooth, via

$$\begin{aligned} \min\ & f_k + \nabla f_k^T d_k + \tfrac{1}{2} d_k^T (H_k + \widetilde{H}_k) d_k \\ \text{s.t.}\ & b_k + \nabla b_k^T d_k \geq 0 \\ & c_k + \nabla c_k^T d_k = 0 \end{aligned}$$

and promote convergence with the ℓ_1 penalty function

$$\phi(x;\rho) \triangleq \rho f(x) + \sum_{i=1}^{l} \max\{0, -b^i(x)\} + \sum_{j=1}^{m} |c^j(x)|.$$

Handling infeasible subproblems

We have various options when a given subproblem is infeasible

- Feasibility restoration: replace the subproblem with

$$\begin{aligned} \min\ & \sum_{i=1}^{l} r^i + \sum_{j=1}^{m} (s^j + t^j) + \tfrac{1}{2} d_k^T \widetilde{H}_k d_k \\ \text{s.t.}\ & b_k + \nabla b_k^T d_k \geq -r \\ & c_k + \nabla c_k^T d_k = -s + t \end{aligned}$$

(... but what if we move away from optimality?)

- Constraint relaxation: replace the subproblem with

$$\begin{aligned} \min\ & \rho \left(f_k + \nabla f_k^T d_k + \tfrac{1}{2} d_k^T H_k d_k \right) + \sum_{i=1}^{l} r^i + \sum_{j=1}^{m} (s^j + t^j) + \tfrac{1}{2} d_k^T \widetilde{H}_k d_k \\ \text{s.t.}\ & b_k + \nabla b_k^T d_k \geq -r \\ & c_k + \nabla c_k^T d_k = -s + t \end{aligned}$$

(... but what if we move away from feasibility?)

Penalty-SQP Capabilities

Sℓ_1QP	Feasible	Infeasible
Global		
Fast Local		

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Penalty-SQP Capabilities

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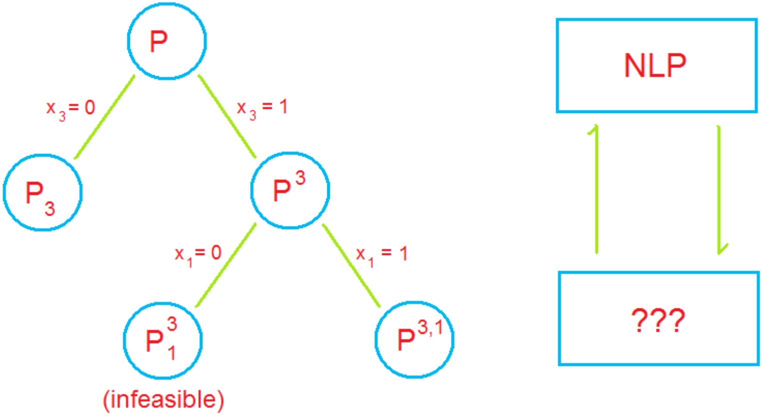
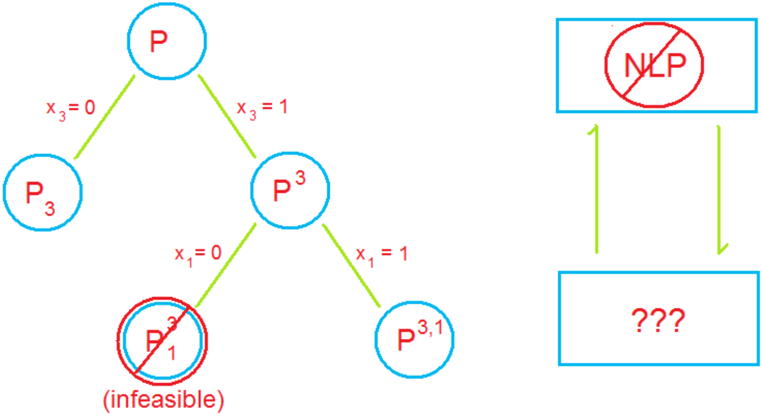
MINLP methods



MINLP methods



Summary

- There is a need for algorithms that converge quickly, regardless of whether the problem is feasible or infeasible
- Interior-point methods are known to perform poorly in infeasible cases, but active set methods seem promising
- Room for improvement in active set methods, too
- Feasibility restoration techniques are an option, but we prefer a single algorithm with no heuristics

A single algorithmic framework

Our goal is to design a *single* optimization algorithm designed to solve

$$(NLP) \triangleq \left\{ \begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & b(x) \geq 0 \\ & c(x) = 0 \end{aligned} \right\}$$

or, if (NLP) is infeasible,

$$(FEAS) \triangleq \left\{ \min\ \ v(x) \triangleq \sum_{i=1}^{l} \max\{0, -b^i(x)\} + \sum_{j=1}^{m} |c^j(x)| \right\}$$

We combine (NLP) and $(FEAS)$ to define

$$(P) \triangleq \{\min\ \ \rho f(x) + v(x)\}$$

where $\rho \geq 0$ is a penalty parameter to be updated dynamically

Our method for step computation and acceptance

We generate a step via

$$\min\ q_k(d_k;\rho)$$

$$\triangleq \rho\left(f_k + \nabla f_k^T d_k + \tfrac{1}{2} d_k^T H_k d_k\right) + \sum_{i=1}^{l} \max\left\{0, -b_k^i - \nabla b_k^{i\,T} d_k\right\} + \sum_{j=1}^{m} \left|c_k^j + \nabla c_k^{j\,T} d_k\right| + \tfrac{1}{2} d_k^T \widetilde{H}_k d_k$$

which is equivalent to the quadratic subproblem

$$(Q_\rho) \triangleq \begin{array}{ll} \min & \rho\left(f_k + \nabla f_k^T d_k + \tfrac{1}{2} d_k^T H_k d_k\right) + \sum_{i=1}^{l} r^i + \sum_{j=1}^{m} (s^j + t^j) + \tfrac{1}{2} d_k^T \widetilde{H}_k d_k \\ \text{s.t.} & b_k + \nabla b_k^T d_k \geq -r \\ & c_k + \nabla c_k^T d_k = -s + t \end{array}$$

and we measure progress with the exact penalty function

$$\phi(x;\rho) \triangleq \rho f(x) + v(x)$$

A Penalty-SQP framework

Step 0. Initialize x_0 and set $\eta \in (0,1)$, $\tau \in (0,1)$ and $k \leftarrow 0$

Step 1. If x_k solves (NLP) or $(FEAS)$, then stop

Step 2. Compute a value for the penalty parameter, call it ρ_k

Step 3. Compute d_k by solving (Q_ρ) with $\rho \leftarrow \rho_k$

Step 4. Let α_k be the first member of the sequence $\{1, \tau, \tau^2, \ldots\}$ s.t.

$$\phi(x_k; \rho_k) - \phi(x_k + \alpha_k d_k; \rho_k) \geq \eta \alpha_k [q_k(0; \rho_k) - q_k(d_k; \rho_k)]$$

Step 5. Update $x_{k+1} \leftarrow x_k + \alpha_k d_k$, go to Step 1

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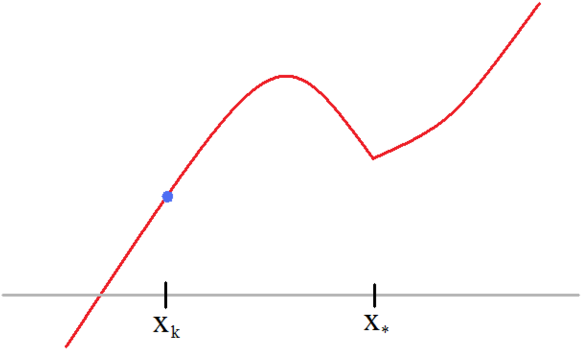
Steering Rules

We extend the *steering rules* for setting ρ_k[1]

1. First, we guarantee first-order progress toward minimizing the feasibility violation measure $v(x)$ during each iteration

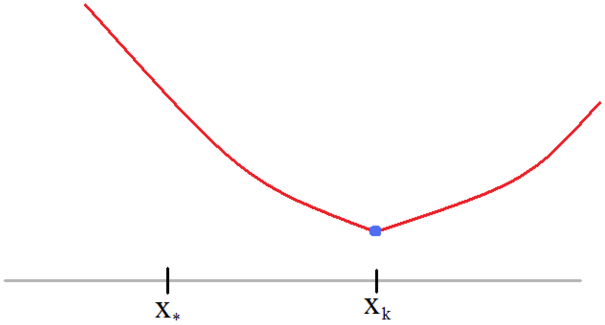


[1] Byrd, Nocedal, Waltz (2008) and Byrd, López-Calva, Nocedal (2008)

Steering Rules

We extend the *steering rules* for setting ρ_k

2. Second, we avoid settling on a penalty function with ρ too large by ensuring that progress in ϕ is proportional to the largest *potential* progress in feasibility



Penalty parameter update

a. If the current iterate is *feasible*, then the algorithm should reduce ρ, if necessary, to ensure linearized feasibility; i.e.,

$$l_k(d_k) \triangleq \sum_{i=1}^{l} \max\left\{0, -b_k^i - \nabla {b_k^i}^T d_k\right\} + \sum_{j=1}^{m} \left|c_k^j + \nabla {c_k^j}^T d\right| = 0$$

b. Otherwise, ρ should be set so that

$$\begin{aligned} l_k(0) - l_k(d_k) &\geq \epsilon_1(l_k(0) - l_k(\bar{d}_k)) \\ q_k(0;\rho_k) - q_k(d_k;\rho_k) &\geq \epsilon_2(q_k(0;0) - q_k(\bar{d}_k;0)) \end{aligned}$$

where the *reference direction* is given by

$$\bar{d}_k \triangleq \arg\min \ (Q_\rho) \text{ for } \rho = 0$$

Penalty parameter update, extended

a. If the current iterate is *feasible*, then the algorithm should reduce ρ, if necessary, to ensure linearized feasibility; i.e.,

$$l_k(d_k) \triangleq \sum_{i=1}^{l} \max\left\{0, -b_k^i - \nabla b_k^{i\,T} d_k\right\} + \sum_{j=1}^{m} \left|c_k^j + \nabla c_k^{j\,T} d\right| = 0$$

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c. Finally, if ρ is decreased we ensure

$$\rho_k \leq \epsilon_3 \|\text{KKT error for } (FEAS)\|^2$$

Strategy for fast convergence

Hitting a moving target:

$$x_k \longrightarrow x_\rho \longrightarrow \hat{x}$$

where

$$\begin{aligned} x_k &\triangleq \text{ } k\text{th iterate of the algorithm} \\ x_\rho &\triangleq \text{ solution of penalty problem } (P) \\ \hat{x} &\triangleq \text{ infeasible stationary point of } (NLP)\text{, solution of } (FEAS) \end{aligned}$$

It has been shown[2] that, for some $C, C' > 0$,

$$\begin{aligned} \|x_{k+1} - \hat{x}\| &\leq \|x_{k+1} - x_\rho\| + \|x_\rho - \hat{x}\| \\ &\leq C\|x_k - x_\rho\|^2 + O(\rho) \\ &\leq C'\|x_k - \hat{x}\|^2 + O(\rho), \end{aligned}$$

so convergence is quadratic if $\rho \propto \|x_k - \hat{x}\|^2$

[2] Byrd, Curtis, Nocedal (2008)

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$x_k \triangleq$ kth iterate of the algorithm

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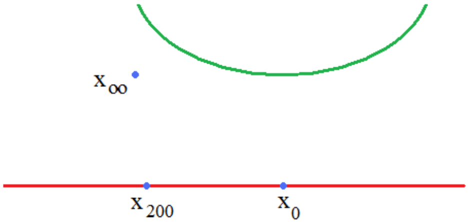
$\hat{x} \triangleq$ infeasible stationary point of (NLP), solution of $(FEAS)$



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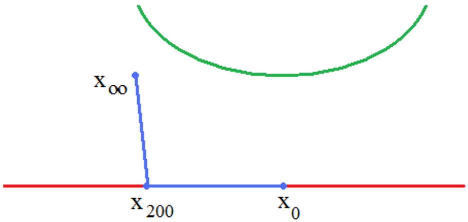
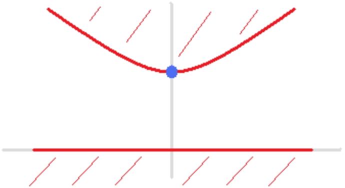


An infeasible problem with a strict minimizer of infeasibility

$$\min \; x_1 + x_2$$
$$\text{s.t. } x_2 - x_1^2 - 1 \geq 0$$
$$0.3(1 - e^{x_2}) \geq 0$$



k	f_k	v_k	$E_k(\rho_{k-1})$	$E_k(0)$	ρ_k	$\|\|d_k\|\|$	α_k
0	+5.00e+00	9.92e+00	8.84e+00	7.51e+00	1.00e+00	1.71e+00	1.00e+00
1	+2.66e+00	2.82e+00	3.51e+00	2.23e+00	1.00e-02	1.16e+00	1.00e+00
2	+1.04e+00	1.06e+00	9.14e-01	9.07e-01	1.00e-02	1.13e+00	5.00e-01
3	+3.20e-01	8.23e-01	4.55e-01	4.53e-01	1.00e-02	7.76e-01	1.00e+00
4	+6.74e-01	6.09e-01	4.33e-01	4.38e-01	1.00e-02	4.66e-01	1.00e+00
5	+9.72e-01	5.81e-01	2.84e-01	2.73e-01	1.00e-02	2.29e-01	1.00e+00
6	+9.39e-01	5.22e-01	4.52e-02	5.49e-02	1.00e-03	4.24e-02	1.00e+00
7	+9.99e-01	5.16e-01	1.50e-03	9.19e-04	8.45e-07	9.42e-04	1.00e+00
8	+1.00e+00	5.15e-01	4.79e-08	8.93e-07	--------	--------	--------

Problem `batch` as a MINLP

k	f_k	v_k	$E_k(\rho_{k-1})$	$E_k(0)$	ρ_k	$\|\|d_k\|\|$	α_k
7	+6.28e+04	1.89e+00	4.16e-02	1.65e+00	1.00e-05	1.50e+00	1.00e+00
8	+9.33e+04	2.88e-01	2.50e-01	3.44e-01	1.00e-05	9.55e-01	1.00e+00
9	+1.03e+05	2.07e-02	1.86e-02	2.44e-02	1.00e-05	2.17e-01	1.00e+00
10	+1.04e+05	4.24e-04	3.59e-04	4.99e-04	2.49e-07	1.58e-03	1.00e+00
11	+1.04e+05	3.76e-08	5.06e-07	--------	--------	--------	--------

Adding the constraint $tl[1] \geq 5$ creates an infeasible problem:

k	f_k	v_k	$E_k(\rho_{k-1})$	$E_k(0)$	ρ_k	$\|\|d_k\|\|$	α_k
11	+2.27e+05	3.09e+00	3.92e-02	3.41e-02	1.00e-06	3.68e-01	1.00e+00
12	+2.27e+05	3.07e+00	3.87e-04	4.32e-02	1.00e-07	7.74e-01	1.00e+00
13	+2.66e+05	3.04e+00	6.76e-03	6.79e-03	1.00e-07	2.24e-01	1.00e+00
14	+2.66e+05	3.04e+00	1.19e-05	1.20e-05	1.44e-10	1.30e-01	1.00e+00
15	+2.66e+05	3.04e+00	7.82e-09	6.02e-09	--------	--------	--------

Summary

- We have argued for the need of optimization algorithms that converge quickly, even on infeasible problems
- We have presented a penalty-SQP approach that transitions smoothly between solving an optimization problem and the corresponding feasibility problem
- The novel components of the algorithm are:
    1. the quadratic model q_k (it is quadratic even for $\rho = 0$)
    2. an appropriate update for ρ to ensure superlinear convergence on infeasible problem instances
- We have illustrated that the algorithm performs well on both feasible and infeasible problem instances

Penalty-SQP Capabilities Now

Thanks!

Sℓ_1QP	Feasible	Infeasible
Global	✓	✓
Fast Local	✓	✓✓✓

Questions?