A Penalty-Interior-Point Algorithm for Nonlinear Optimization

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International Conference on Continuous Optimization 2010

July 27, 2010

Based on "A Penalty-Interior-Point Method for Large-Scale Nonlinear Optimization," submitted for publication in Mathematical Programming, 2010.

Outline

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Large-scale optimization

Consider the optimization problem:

$$(\text{OP}) := \boxed{\begin{array}{l} \min_{x} \ f(x) \\ \text{s.t. } c(x) \leq 0 \end{array}}$$

For large-scale instances:

- Linear or quadratic optimization subproblems are expensive. (Linear systems OK.)
- The constraints may be difficult to satisfy.
- The constraints may be (locally) infeasible; i.e., the algorithm should solve:

$$(\text{FP}) := \boxed{\min_{x} \ v(x) := \sum_{i \in \mathcal{I}} \max\{c^i(x), 0\}}$$

Penalty methods

Unconstrained techniques can be used if we solve:

$$\boxed{\min_{x} \ \rho f(x) + v(x)}$$

Similarly, we can solve a regularized form of (OP):

$$(\text{PP}) := \boxed{\begin{array}{l} \min\limits_{x,s} \ \rho f(x) + \sum\limits_{i \in \mathcal{I}} s^i \\ \text{s.t. } c(x) - s \leq 0, \ s \geq 0 \end{array}}$$

- Unconstrained techniques may fail or be slow if f is unbounded below; performance depends greatly on the form of v.
- Solving (PP) commonly requires the solution of linear or quadratic subproblems.
- Either way, updating the penalty parameter is a challenge.

Interior-point methods

Large-scale problems are often solved efficiently through interior-point subproblems:

$$
(\text{IP}) := \boxed{\begin{aligned} &\min_{x,r} \; f(x) - \mu \sum_{i \in \mathcal{I}} \ln r^i \\ &\text{s.t. } c(x) + r = 0 \quad (\text{with } r > 0) \end{aligned}}
$$

- Lacks constraint regularization as in a penalty method.
- Similar to before, updating the interior-point parameter is a challenge.

Penalty-interior-point methods(?)

Can penalty and interior-point ideas be combined to create a practical algorithm?

- Regularization through penalties is an attractive feature.
- Search direction computations via linear system solves is nice for large problems.

However, there are significant challenges:

- Penalty methods want the algorithm to be free to violate constraints.
- Interior-point methods want the algorithm to remain feasible.
- Juggling "conflicting" parameters is a major challenge.

Literature

Previous work with similar motivations:

- Jittorntrum and Osborne (1980)
- Polyak (1982, 1992, 2008)
- Breitfeld and Shanno (1994, 1996)
- Goldfarb, Polyak, Scheinberg, and Yuzefovich (1999)
- Gould, Orban, and Toint (2003)
- Chen and Goldfarb (2006, 2006)
- Benson, Sen, and Shanno (2008)

However, in my view, none of these papers focus enough on parameter updates.

Outline

Introduction

Algorithmic Framework

Parameter Updates

Numerical Experiments

Summary and Future Work

Penalty-interior-point subproblem

Recall:

$$(\text{PP}) := \boxed{\begin{aligned} &\min_{x,s}\ \rho f(x) + \sum_{i\in\mathcal{I}} s^i \\ &\text{s.t. } c(x) - s \le 0,\ s \ge 0 \end{aligned}} \qquad (\text{IP}) := \boxed{\begin{aligned} &\min_{x,r}\ f(x) - \mu \sum_{i\in\mathcal{I}} \ln r^i \\ &\text{s.t. } c(x) + r = 0 \quad (\text{with } r > 0) \end{aligned}}$$

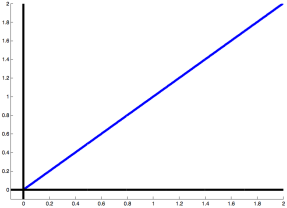
Applying an interior-point reformulation to (PP), we can obtain:

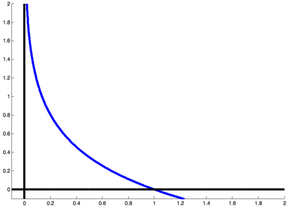
$$(\text{PIP}) := \boxed{\begin{aligned} &\min_{x,r,s}\ \rho f(x) - \mu \sum_{i\in\mathcal{I}} (\ln r^i + \ln s^i) + \sum_{i\in\mathcal{I}} s^i \\ &\text{s.t. } c(x) + r - s = 0 \quad (\text{with } r, s > 0) \end{aligned}}$$

- (PIP) satisfies MFCQ (it is a reformulation of (PP), which also satisfies it).
- $\mu \to 0$ and $\rho \to \bar{\rho} > 0$ to obtain a solution to (OP).
- $\mu \to 0$ and $\rho \to 0$ to obtain a solution to (FP).

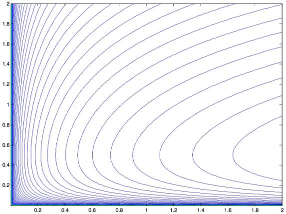
Visualizing the penalty-interior-point objective

- Objective function terms for s^i in (PP) and r^i in (IP):





- Objective function term for (r^i, s^i) in (PIP):



Algorithm outline

for $k = 0, 1, 2, \ldots$

- Reset the slack variables.
- Update the parameters.
- Compute a search direction.
- Perform a line search.

Slack reset

Through the slack variables, we have added many degrees of freedom to the problem!

- However, for a fixed x_k, (PIP) reduces to

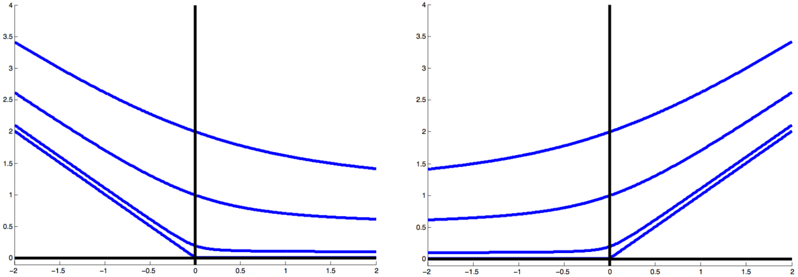
$$\min_{r,s} \; -\mu \sum_{i \in \mathcal{I}} (\ln r^i + \ln s^i) + \sum_{i \in \mathcal{I}} s^i$$
$$\text{s.t. } c(x_k) + r - s = 0 \quad (\text{with } r, s > 0)$$

- This problem is convex and separable, and has the unique solution:

$$r_k^i = r^i(x_k;\mu) := \mu - \tfrac{1}{2} c^i(x_k) + \tfrac{1}{2}\sqrt{c^i(x_k)^2 + 4\mu^2}$$
$$\text{and} \quad s_k^i = s^i(x_k;\mu) := \mu + \tfrac{1}{2} c^i(x_k) + \tfrac{1}{2}\sqrt{c^i(x_k)^2 + 4\mu^2}.$$

Visualizing the slack reset

Slack variables r and s, respectively, as functions of μ and $c(x_k)$:



Search direction calculation

A Newton iteration for the optimality conditions of (PIP) involves:

$$\begin{bmatrix} H_k & 0 & 0 & \nabla c(x_k) \\ 0 & \Omega_k & 0 & I \\ 0 & 0 & \Gamma_k & -I \\ \nabla c(x_k)^T & I & -I & 0 \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta r_k \\ \Delta s_k \\ \Delta \lambda_k \end{bmatrix} = - \begin{bmatrix} \rho \nabla f(x_k) + \nabla c(x_k) \lambda_k \\ \lambda_k - \mu R_k^{-1} e \\ e - \lambda_k - \mu S_k^{-1} e \\ c(x_k) + r_k - s_k \end{bmatrix}$$

(Why do we still have $(\Delta r_k, \Delta s_k)$ if we eliminated (r_k, s_k)?)

Merit function

- Recall that the objective of (PIP) is given by

$$\phi(x, r, s; \rho, \mu) := \rho f(x) - \mu \sum_{i \in \mathcal{I}} (\ln r^i + \ln s^i) + \sum_{i \in \mathcal{I}} s^i.$$

- A standard type of merit function or filter for (PIP) would involve ϕ and a measure of violation of the constraints $c(x) + r - s = 0$.
- However, the slack reset allows us to use the merit function

$$\widetilde{\phi}(x; \rho, \mu) := \rho f(x) - \mu \sum_{i \in \mathcal{I}} (\ln r^i(x; \mu) + \ln s^i(x; \mu)) + \sum_{i \in \mathcal{I}} s^i(x; \mu).$$

Lemma

Let $r_k = r(x_k; \mu)$ and $s_k = s(x_k; \mu)$. Then, the computed search direction Δx_k yielded by the Newton system is a descent direction for $\widetilde{\phi}(x; \rho, \mu)$ at $x = x_k$.

Line search

For a given search direction $(\Delta x_k, \Delta\lambda_k)$, we:

- backtrack to find $\alpha_k \in (0,1]$ satisfying the fraction-to-the-boundary rules

$$\begin{aligned} r(x_k + \alpha\Delta x_k; \mu) &\geq \tau r_k \\ \text{and} \quad s(x_k + \alpha\Delta x_k; \mu) &\geq \tau s_k \end{aligned}$$

  and the sufficient decrease condition

$$\widetilde{\phi}(x_k + \alpha_k \Delta x_k; \rho, \mu) \leq \widetilde{\phi}(x_k; \rho, \mu) + \eta\alpha_k \nabla\widetilde{\phi}(x_k; \rho, \mu)^T \Delta x_k.$$

- compute the largest $\beta_k \in (0,1]$ satisfying the fraction-to-the-boundary rule

$$\lambda_k + \beta\Delta\lambda_k \in [\tau\lambda_k, e - \tau(e - \lambda_k)].$$

Outline

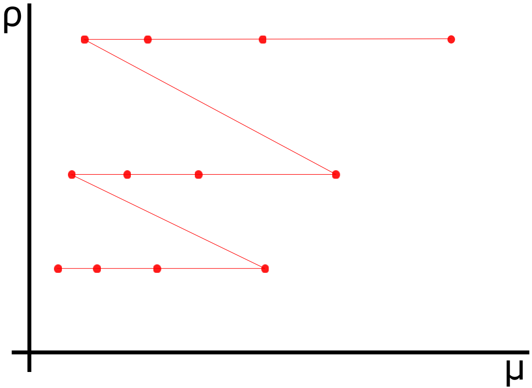
Conservative strategies

A simple, conservative strategy may be the following:

- Step 1: Fix ρ and solve (PIP) for $\mu \to 0$.
- Step 2: If we are infeasible, decrease ρ and go to step 1.

This strategy, or ones that are equally as conservative, are the type that have been implemented in many other penalty-interior-point algorithms.

Visualizing a conservative strategy



- Each "dot" may require at least a few iterations.
- Each "row" may require the computational effort of an entire interior-point run!
- For an infeasible problem, we need both ρ and μ to reduce to (near) zero.

Search direction calculation

Recall the Newton system:

$$\begin{bmatrix} H_k & 0 & 0 & \nabla c(x_k) \\ 0 & \Omega_k & 0 & I \\ 0 & 0 & \Gamma_k & -I \\ \nabla c(x_k)^T & I & -I & 0 \end{bmatrix} \begin{bmatrix} \Delta x_k \\ \Delta r_k \\ \Delta s_k \\ \Delta \lambda_k \end{bmatrix} = - \begin{bmatrix} \rho \nabla f(x_k) + \nabla c(x_k) \lambda_k \\ \lambda_k - \mu R_k^{-1} e \\ e - \lambda_k - \mu S_k^{-1} e \\ 0 \end{bmatrix}$$

- ρ and μ may be embedded in H_k, Ω_k, and Γ_k, but...
- Holding these matrices fixed for iteration k means we have a system of the form:

$$M \Delta z_k^{\rho,\mu} = \rho \begin{bmatrix} -\nabla f(x_k) \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ R_k^{-1} e \\ S_k^{-1} e \\ 0 \end{bmatrix} + \begin{bmatrix} -\nabla c(x_k)^T \lambda_k \\ -\lambda_k \\ -e + \lambda_k \\ 0 \end{bmatrix}.$$

- Thus, the solution for all pairs (ρ, μ) can be obtained with only one factorization.

Update criteria

- It is computationally practical to vary ρ and μ on the right-hand side.
- Ok, but what criteria should we use for choosing these values?
- In a penalty method, decreasing ρ places more emphasis on the constraints.
- In an interior-pont method, decreasing μ places less emphasis on centrality.
- However, in a penalty-interior-point method, everything gets jumbled!

In short, we update:

- ρ to ensure some level of progress toward solving the primal feasibility problem;
- μ to attempt to satisfy dual feasibility and complementarity.

A basis for comparison

Let $z := (x, r, s)$.

- We have two views of the penalty-interior-point objective:

$$\phi(z; \rho, \mu) = \rho f(x) - \mu \sum_{i \in \mathcal{I}} (\ln r^i + \ln s^i) + \sum_{i \in \mathcal{I}} s^i;$$
$$\widetilde{\phi}(x; \rho, \mu) = \rho f(x) - \mu \sum_{i \in \mathcal{I}} (\ln r^i(x; \mu) + \ln s^i(x; \mu)) + \sum_{i \in \mathcal{I}} s^i(x; \mu).$$

- Thus, we have two corresponding linear models:

$$l(\Delta z; \rho, \mu, z) := \phi(z; \rho, \mu) + \nabla \phi(z; \rho, \mu)^T \Delta z;$$
$$\widetilde{l}(\Delta x; \rho, \mu, x) := \widetilde{\phi}(x; \rho, \mu) + \nabla \widetilde{\phi}(x; \rho, \mu)^T \Delta x.$$

- For the μ used in the slack reset, the models coincide, but not otherwise.
- It is easily seen in the direction computation that

$$\Delta l(\Delta z; \rho, \mu, z) := l(0; \rho, \mu, z) - l(\Delta z; \rho, \mu, z) > 0,$$

but it is a reduction in $\widetilde{l}(\cdot; \rho, \mu, x)$ that we want to guarantee!

Updating ρ: ensuring progress in feasibility

Let $\Delta z_k^{\rho,\mu}$ be the direction computed for given (ρ, μ).

- If x_k is feasible, then choose largest ρ such that for some μ:

$$\Delta\widetilde{q}(\Delta x_k^{\rho,\mu}; \rho, \mu, x_k) > 0.$$

  (Here, $\widetilde{q}(\Delta x; \rho, \mu, x)$ is a quadratic model of $\widetilde{\phi}(x; \rho, \mu)$.)

- If x_k is infeasible, then choose largest ρ such that for some μ:

$$\begin{aligned} \Delta\widetilde{l}(\Delta x_k^{\rho,\mu}; \rho, \mu, x_k) &\geq \epsilon_1 \Delta l(\Delta z_k^{0,\mu}; 0, \mu, z_k), \quad \epsilon_1 \in (0,1); \\ \Delta\widetilde{q}(\Delta x_k^{\rho,\mu}; \rho, \mu, x_k) &\geq \epsilon_2 \Delta l(\Delta z_k^{0,\mu}; 0, \mu, z_k), \quad \epsilon_2 \in (0,1). \end{aligned}$$

Updating ρ: promoting fast infeasibility detection

If x_k is infeasible, then ρ must satisfy

$$\rho \leq \left\| \begin{bmatrix} \nabla c(x_k)\lambda_k \\ R_k\lambda_k \\ S_k(e-\lambda_k) \end{bmatrix} \right\|^2$$

(Right-hand side only small in neighborhood of an infeasible stationary point.)

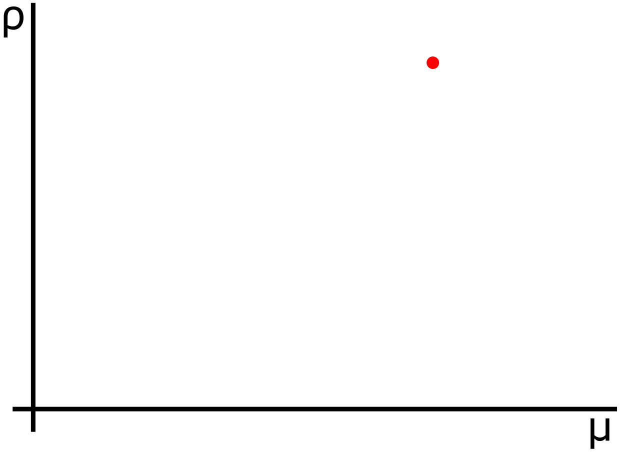
Updating μ: minding dual feasibility and complementarity

Fixing ρ, now choose μ so that the previous conditions still hold and

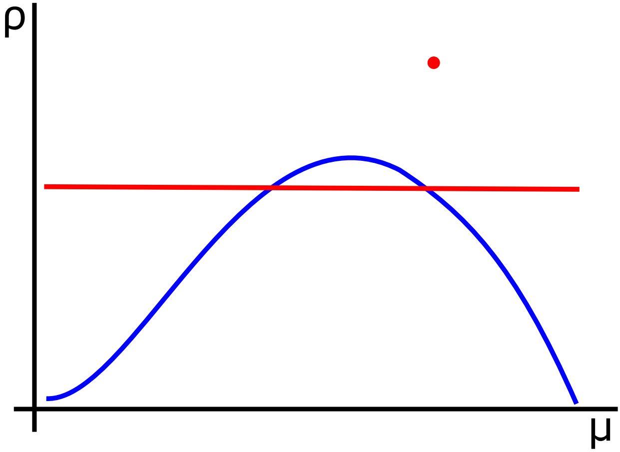
$$m(\Delta z, \Delta\lambda; \rho, \mu, z_k) := \left\| \begin{bmatrix} \rho\nabla f(x_k) + \nabla c(x_k)(\lambda_k + \Delta\lambda) \\ (R_k + \Delta R)(\lambda_k + \Delta\lambda) \\ (S_k + \Delta S)(e - \lambda_k - \Delta\lambda) \end{bmatrix} \right\|_\infty$$

is approximately minimized.

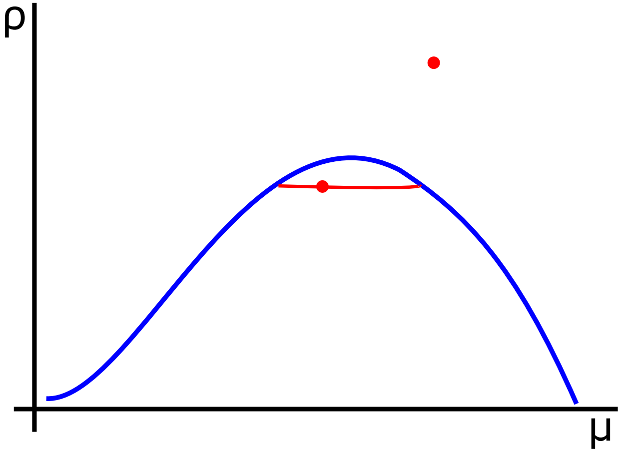
Visualizing the aggressive strategy



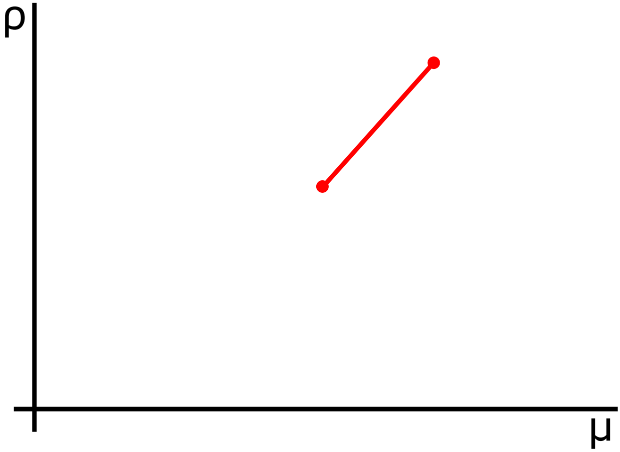
Visualizing the aggressive strategy



Visualizing the aggressive strategy



Visualizing the aggressive strategy



All within one iteration!

Outline

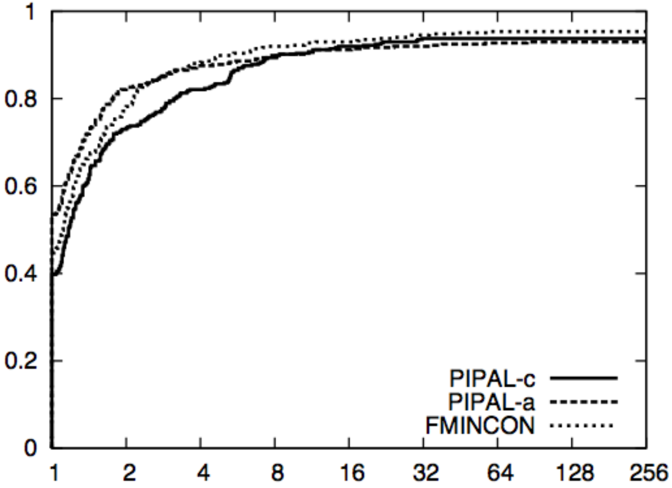
Preliminary experimentation

- Penalty-Interior-Point ALgorithm (PIPAL)
- Compared iteration counts for PIPAL-c, PIPAL-a, and FMINCON.[1]
- CUTEr problems available in AMPL (385 in final set)
- Infeasible variants of the HS problems (93 in final set):

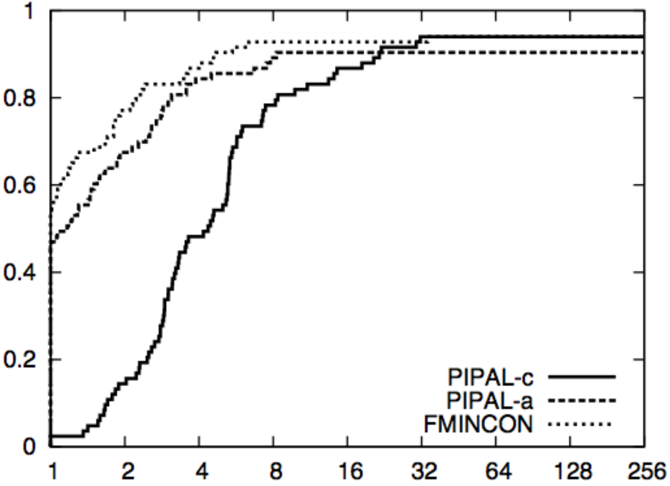
$$c^I(x) = 0 \Rightarrow \{c^I(x) = 0 \;\&\; c^I(x) = 1\}$$
$$c^I(x) \le 0 \Rightarrow \{c^I(x) \le 0 \;\&\; c^I(x) \ge 1\}$$

[1] See Waltz, Morales, Nocedal, and Orban (2006)

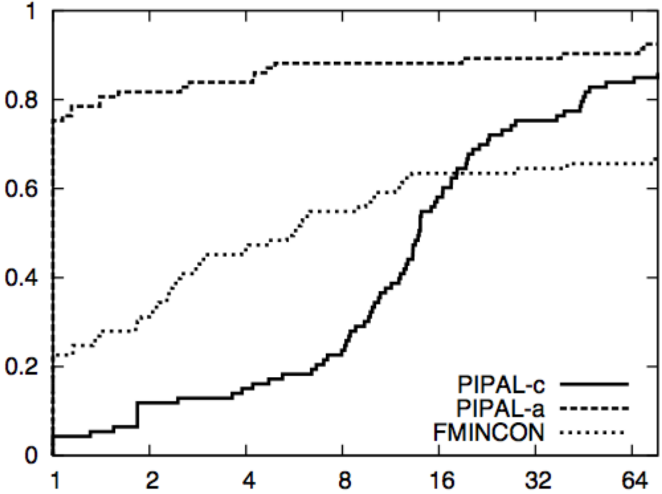
Entire test set



Problems requiring a penalty parameter decrease



Infeasible problems



Outline

Summary

- Combined penalty and interior-point techniques into a single algorithm.
- Penalties effectively regularize the constraints.
- Interior-point strategies allowed directions to be computed via linear systems.
- Slack reset allowed us to reduce the degrees of freedom.
- Proposed an aggressive updating scheme for the parameters.
- Results are comparable to an interior-point method on most problems.
- Results are much better than an interior-point method on infeasible problems.

Future work

- Advanced implementation is required — numerical issues are critical!
- Parameter update strategy for MPCCs — needs to be different?
- Specializations for MINLP — especially in terms of infeasibility detection?

A new normal step?

- Our update for ρ involved comparing directions to that obtained with $\rho = 0$.
- That is, it is compared to an interior-point step for the feasibility problem:

$$(\text{FIP}) := \boxed{\begin{aligned} \min_{\Delta x, \Delta r, \Delta s} \quad & -\mu(R_k^{-1}\Delta r + S_k^{-1}\Delta s) + \sum_{i \in \mathcal{I}} \Delta s^i \\ \text{s.t.} \quad & (c(x_k) + \nabla c(x_k)^T \Delta x) + (r_k + \Delta r) - (s_k + \Delta s) = 0 \end{aligned}}$$

- In contrast, step decomposition methods attempt to solve:

$$(\text{NP}) := \boxed{\begin{aligned} \min_{\Delta x, \Delta s} \quad & \|c(x_k) + \nabla c(x_k)^T \Delta x - (s_k + S_k(S_k^{-1}\Delta s))\|^2 \\ \text{s.t.} \quad & \|\Delta x\|^2 + \|S_k^{-1}\Delta s\|^2 \le \delta_k^2 \\ & s_k + \Delta s \ge \tau s_k \end{aligned}}$$

- Does PIPAL offer an alternative to the standard normal subproblem?