

A Sequential Algorithm for Solving Nonlinear Optimization Problems With Chance Constraints

Frank E. Curtis, Lehigh University

joint work with

Andreas Wächter, Northwestern University
Victor M. Zavala, University of Wisconsin–Madison

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Problem Statements

Penalty Function

Algorithm

Experiments

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Chance-constrained optimization

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t. } \mathbb{P}_{\xi}[c(x, \xi) \leq 0] \geq 1 - \alpha$$

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$c : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^m$$

- ▶ Random variable ξ with associated space (Ω, \mathcal{F}, P)
- ▶ Assume f and $c(\cdot, \xi)$ are \mathcal{C}^1 for any realization of ξ
- ▶ Even if f and c are linear, the feasible region might not be convex
- ▶ Assume $m = 1$ (though $m > 1$ or multiple chance constraints also OK)

Cardinality-constrained optimization

- ▶ Sample Average Approximation (SAA):

$\Omega = \{\xi_1, \dots, \xi_N\}$ with equal probability

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t. } |\{\xi_i \in \Omega : c(x, \xi_i) \leq 0\}| \geq \lceil (1 - \alpha)N \rceil$$

(CCP)

Cardinality-constrained optimization

- ▶ Sample Average Approximation (SAA):

$\Omega = \{\xi_1, \dots, \xi_N\}$ with equal probability

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t. } |\{i : c_i(x) \leq 0\}| \geq M$$

(CCP)

... arise in other applications as well

Contribution

Propose an SQP-type method for solving problem CCP

- ▶ Viable approach even when, e.g., MINLP techniques are intractable
- ▶ Like penalty-SQP, sequential minimization of penalty function
- ▶ Novel penalty function with “exactness” properties
- ▶ Subproblems with linear chance constraints
 - ▶ Do not need to “predict” optimal subset of constraints
 - ▶ Efficient methods, e.g. [Luedtke 14], [Küçükyavuz 12]
- ▶ Global convergence guarantees
- ▶ Good performance on test problems

Outline

Problem Statements

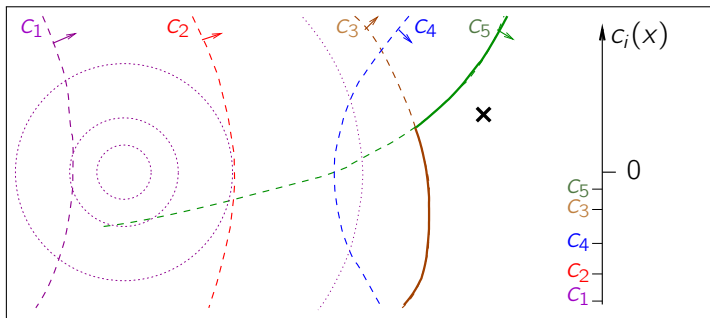
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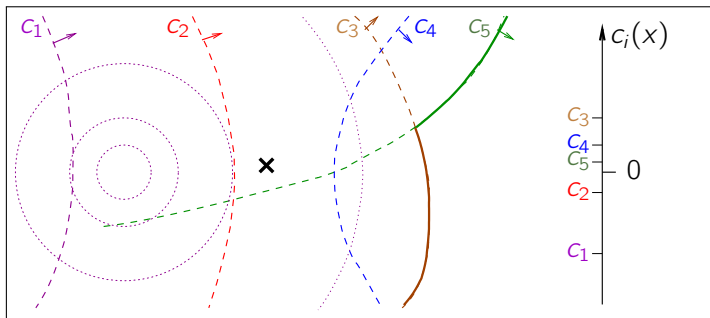
Exact penalty function for deterministic optimization



$$\begin{aligned}\phi(x) &= f(x) + \rho \|[c(x)]_+\|_1 \\ &= f(x) + \rho \sum_{i=1}^N \max\{0, c_i(x)\}\end{aligned}$$

Under CQ, local mins of ϕ correspond to local mins of optimization problem

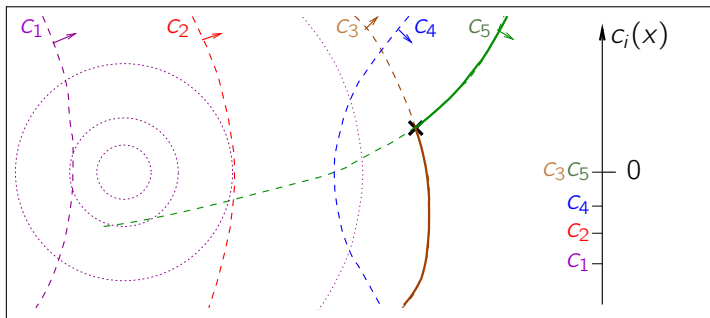
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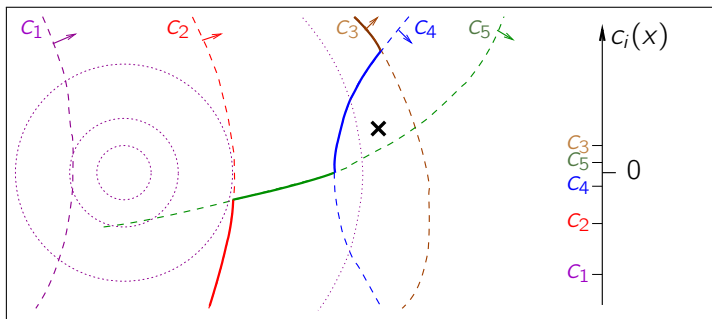
Exact penalty function for deterministic optimization



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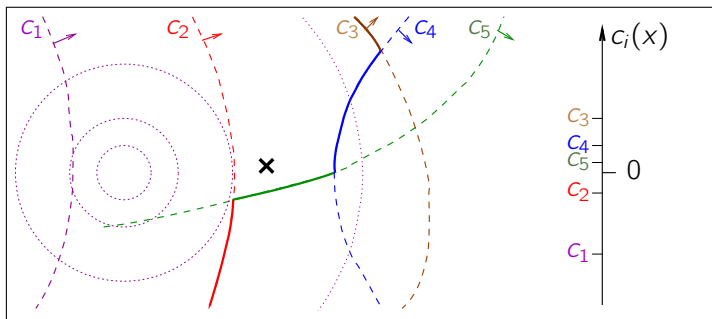
Under CQ, local mins of ϕ correspond to local mins of optimization problem

Constraint violation measure for CCP



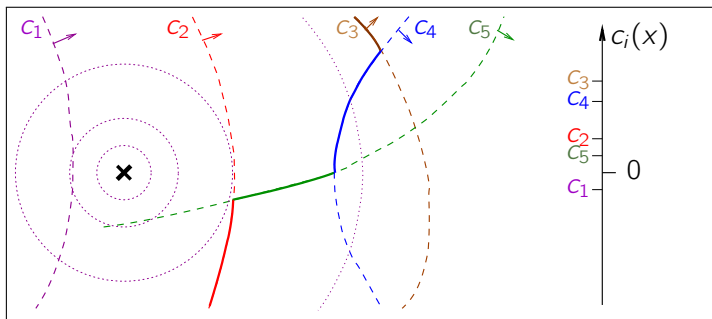
- ▶ Order constraints: $c_{i(1)}(x) \leq c_{i(2)}(x) \leq \dots$
- ▶ Violation measure: $\langle\langle [c(x)]_+ \rangle\rangle_M = \sum_{k=1}^M \max\{0, c_{i(k)}(x)\}$
- ▶ Here: $\langle\langle [c(x)]_+ \rangle\rangle_M = 0$

Constraint violation measure for CCP



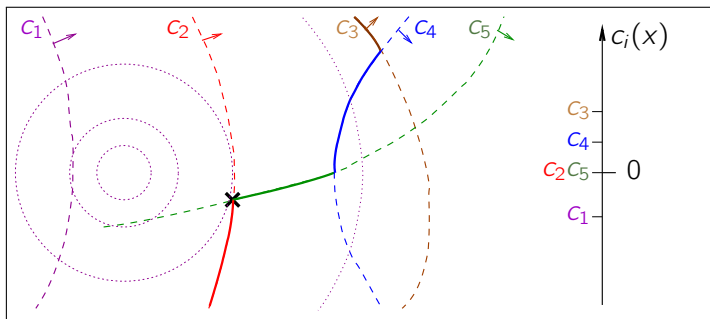
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- ▶ Here: $\langle\langle [c(x)]_+ \rangle\rangle_M = c_5(x)$

Constraint violation measure for CCP



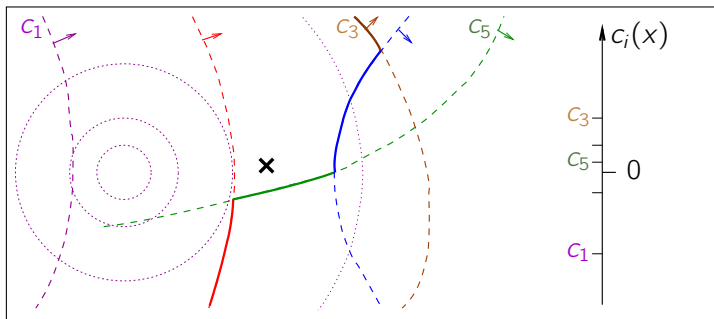
- ▶ Order constraints: $c_{i(1)}(x) \leq c_{i(2)}(x) \leq \dots$
- ▶ Violation measure: $\langle\langle [c(x)]_+ \rangle\rangle_M = \sum_{k=1}^M \max\{0, c_{i(k)}(x)\}$
- ▶ Here: $\langle\langle [c(x)]_+ \rangle\rangle_M = c_2(x) + c_5(x)$

Constraint violation measure for CCP



- ▶ Order constraints: $c_{i(1)}(x) \leq c_{i(2)}(x) \leq \dots$
- ▶ Violation measure: $\langle\langle [c(x)]_+ \rangle\rangle_M = \sum_{k=1}^M \max\{0, c_{i(k)}(x)\}$
- ▶ Here: $\langle\langle [c(x)]_+ \rangle\rangle_M = 0$

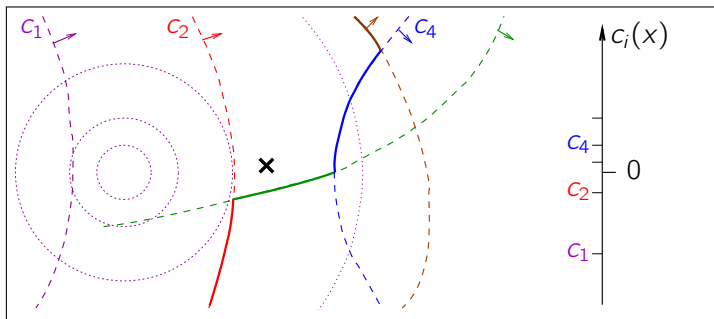
Equivalent formulation



$$\mathfrak{S} := \{\mathcal{S} \subseteq \{1, \dots, N\} : |\mathcal{S}| = M\} \quad c_{\mathcal{S}}(x) := [c_i(x)]_{i \in \mathcal{S}}$$

- ▶ Consider $\|[c_{\mathcal{S}}(x)]_+\|_1$ for different $\mathcal{S} \in \mathfrak{S}$
- ▶ Here: $\mathcal{S} = \{1, 3, 5\}$: $\|[c_{\mathcal{S}}(x)]_+\|_1 = c_5(x) + c_3(x)$

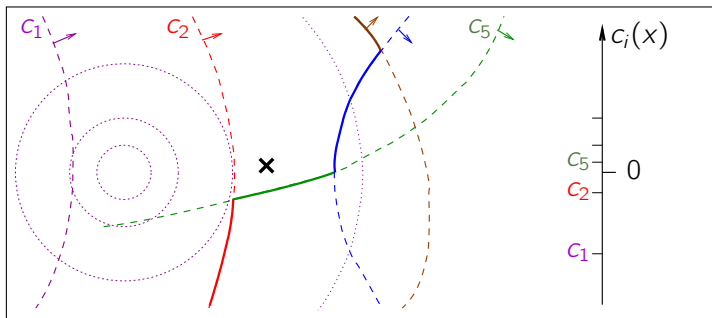
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- ▶ Here: $\mathcal{S} = \{1, 2, 4\}$: $\|[c_{\mathcal{S}}(x)]_+\|_1 = c_4(x)$

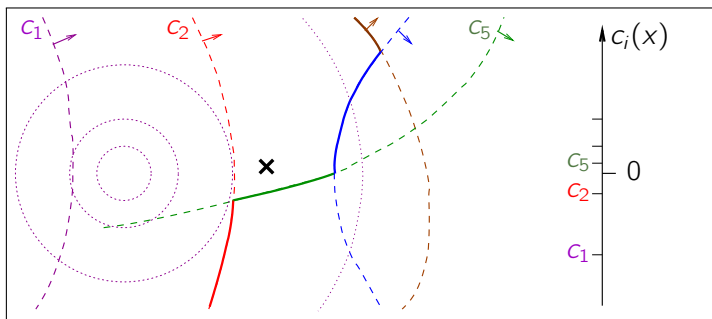
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$$\mathfrak{S} := \{\mathcal{S} \subseteq \{1, \dots, N\} : |\mathcal{S}| = M\} \quad c_{\mathcal{S}}(x) := [c_i(x)]_{i \in \mathcal{S}}$$

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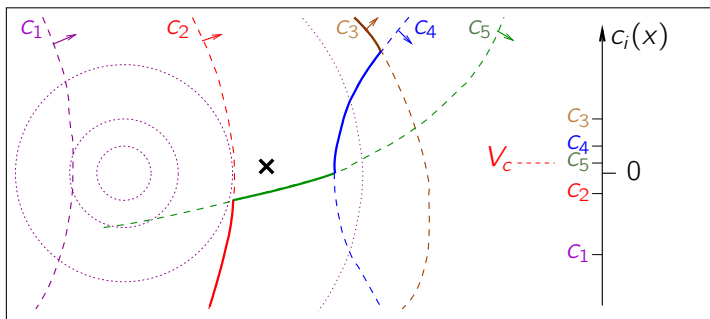
Equivalent formulation



$$\mathfrak{S} := \{\mathcal{S} \subseteq \{1, \dots, N\} : |\mathcal{S}| = M\} \quad c_{\mathcal{S}}(x) := [c_i(x)]_{i \in \mathcal{S}}$$

- ▶ Consider $\|[c_{\mathcal{S}}(x)]_+\|_1$ for different $\mathcal{S} \in \mathfrak{S}$
- ▶ So $\langle\langle [c(x)]_+ \rangle\rangle_M = \min_{\mathcal{S} \in \mathfrak{S}} \|[c_{\mathcal{S}}(x)]_+\|_1$

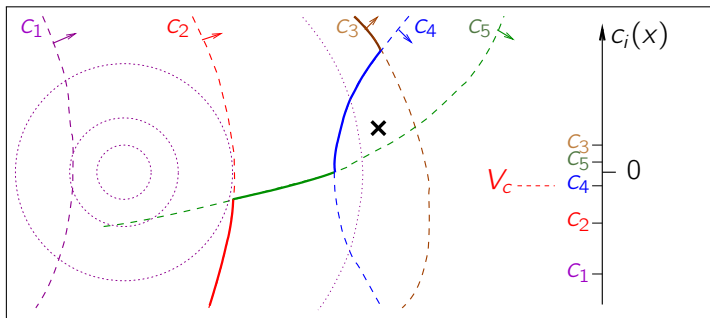
Critical scenario selections



$$\mathfrak{S}(x) = \{\mathcal{S} \in \mathfrak{S} : c_i(x) \leq V_c(x) \text{ for } i \in \mathcal{S}\} \quad V_c(x) = c_{i(M)}(x)$$

- ▶ Then $\| [c(x)]_+ \|_M = \| [c_{\mathcal{S}}(x)]_+ \|_1$ for all $\mathcal{S} \in \mathfrak{S}(x)$.
- ▶ Here: $\mathfrak{S}(x) = \{(1, 2, 5)\}$

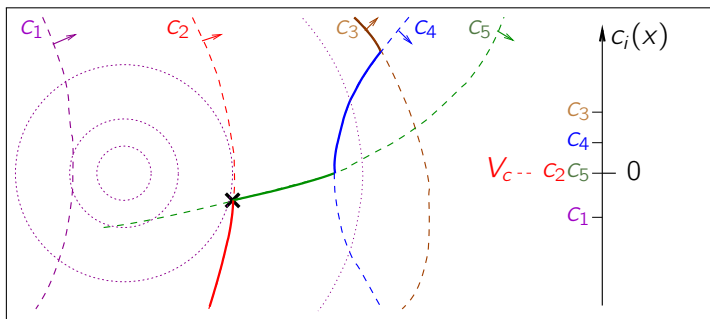
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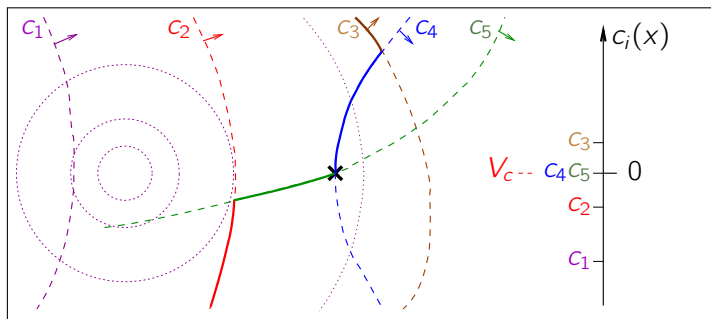
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Critical scenario selections



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- ▶ Here: $\mathfrak{S}(x) = \{(1, 2, 4), (1, 2, 5)\}$

Exact penalty function for CCP

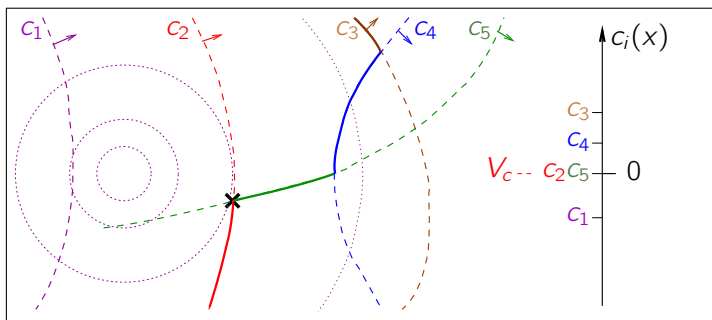
Penalty function for deterministic optimization with constraints in $\mathcal{S} \in \mathfrak{S}$:

$$\phi_{\mathcal{S}}(x) := f(x) + \rho \|[c_{\mathcal{S}}(x)]_+\|_1$$

Penalty function for CCP:

$$\begin{aligned} \phi(x) &= f(x) + \rho \langle\langle [c(x)]_+ \rangle\rangle_M \\ &= f(x) + \rho \min_{\mathcal{S} \in \mathfrak{S}} \|[c_{\mathcal{S}}(x)]_+\|_1 \\ &= \min_{\mathcal{S} \in \mathfrak{S}} f(x) + \rho \|[c_{\mathcal{S}}(x)]_+\|_1 \\ &= \min_{\mathcal{S} \in \mathfrak{S}} \phi_{\mathcal{S}}(x) \\ &= \phi_{\tilde{\mathcal{S}}}(x) \quad (\text{for all } \tilde{\mathcal{S}} \in \mathfrak{S}(x)) \end{aligned}$$

Equivalence of minimizers

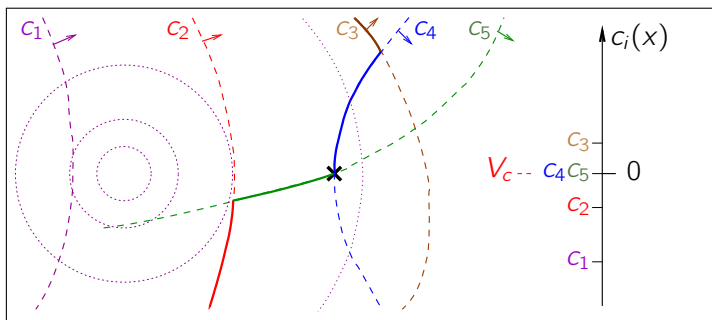


For $\rho > 0$ sufficiently large (and some CQ):

$$\begin{aligned}
 & x_* \text{ local min of CCP} \\
 & \quad \updownarrow \\
 & \llbracket [c(x_*)]_+ \rrbracket_M = 0 \text{ and } x_* \text{ local min of } \phi_{\mathcal{S}}(x) \text{ for all } \mathcal{S} \in \mathfrak{S}(x_*)
 \end{aligned}$$

- Here $\mathfrak{S}(x_*) = \{(1, 2, 5)\}$ and x_* is local min of CCP

Equivalence of minimizers



For $\rho > 0$ sufficiently large (and some CQ):

$$\begin{aligned}
 & x_* \text{ local min of CCP} \\
 & \quad \updownarrow \\
 & \langle\langle [c(x_*)]_+ \rangle\rangle_M = 0 \text{ and } x_* \text{ local min of } \phi_S(x) \text{ for all } S \in \mathfrak{S}(x_*)
 \end{aligned}$$

- Here $\mathfrak{S}(x_*) = \{(1, 2, 4), (1, 2, 5)\}$ and x_* is NOT local min of CCP

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Trust-region method for deterministic optimization ($S\ell_1$ QP)

$$\begin{aligned}\phi(x) &= f(x) + \rho\|[c(x)]_+\|_1 & (g_k = \nabla f(x_k), J_k = \nabla c(x_k)^T) \\ m_k(d) &= f_k + g_k^T d + \frac{1}{2}d^T H_k d + \rho\|[c_k + J_k d]_+\|_1\end{aligned}$$

At iterate x_k :

1. Compute step d_k from trust region subproblem (QP):

$$\min_{d \in \mathbb{R}^n} m_k(d) \quad \text{s.t.} \quad \|d\|_\infty \leq \delta_k$$

2. Define ratio

$$r_k = \frac{\phi(x_k) - \phi(x_k + d_k)}{m_k(0) - m_k(d_k)}$$

3. Update iterate and trust region radius

$$(\mu \in (0, 1))$$

- ▶ If $r_k \geq \mu$: $x_{k+1} \leftarrow x_k + d_k$ and $\delta_{k+1} \leftarrow 2\delta_k$
- ▶ If $r_k < \mu$: $x_{k+1} \leftarrow x_k$ and $\delta_{k+1} \leftarrow 0.5\|d_k\|_\infty$

Trust-region method for CCP

$$\phi(x) = f(x) + \rho \langle [c(x)]_+ \rangle_M$$

$$m_k(d) = f_k + g_k^T d + \frac{1}{2} d^T H_k d + \rho \langle [c_k + J_k d]_+ \rangle_M$$

At iterate x_k :

1. Compute step d_k from trust region subproblem (QCCP):

$$\min_{d \in \mathbb{R}^n} m_k(d) \quad \text{s.t.} \quad \|d\|_\infty \leq \delta_k$$

2. Define ratio

$$r_k = \frac{\phi(x_k) - \phi(x_k + d_k)}{m_k(0) - m_k(d_k)}$$

3. Update iterate and trust region radius

- ▶ If $r_k \geq \mu$: $x_{k+1} \leftarrow x_k + d_k$ and $\delta_{k+1} \leftarrow \max\{2\delta_k, \delta_{\text{reset}}\}$
- ▶ If $r_k < \mu$: $x_{k+1} \leftarrow x_k$ and $\delta_{k+1} = 0.5\|d_k\|_\infty$

$$(\mu \in (0, 1))$$

$$(\delta_{\text{reset}} > 0)$$

Solving the trust region subproblem

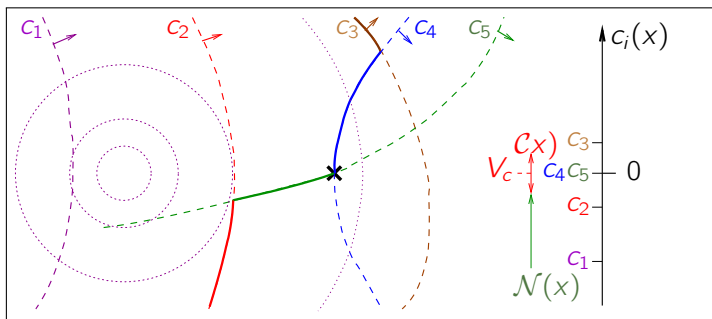
$$\begin{aligned} m_k(d) &= f_k + g_k^T d + \frac{1}{2} d^T H_k d + \rho \langle\langle [c_k + J_k d]_+ \rangle\rangle_M \\ &= f_k + g_k^T d + \frac{1}{2} d^T H_k d + \rho \min_{s \in \mathfrak{S}} \|[c_{S,k} + J_{S,k} d]_+\|_1 \end{aligned}$$

MIQP formulation:

$$\begin{aligned} \min_{d,s,z} \quad & f_k + g_k^T d + \frac{1}{2} d^T H_k d + \rho \sum_{i=1}^N s_i \\ \text{s.t.} \quad & c_i(x_k) + \nabla c_i(x_k)^T d \leq s_i + \text{bigM}(1 - z_i) \\ & \sum_{i=1}^N z_i = M, \quad s \geq 0, \quad \|d\|_\infty \leq \delta_k, \quad z \in \{0, 1\}^N \end{aligned}$$

- ▶ big-M formulation is computationally expensive
- ▶ Better alternative: Use B&C method of [Luedtke 14]
- ▶ Can we reduce the number of scenarios to consider?

ϵ -critical scenario selections



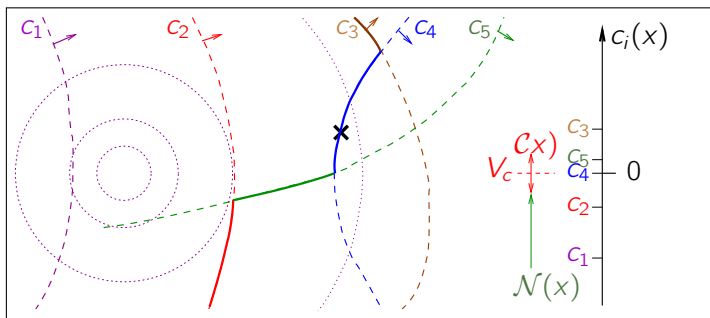
$$\mathcal{C}(x, \epsilon) = \{i \in \{1, \dots, N\} : |c_i(x) - V_c(x)| \leq \epsilon\}$$

$$\mathcal{N}(x, \epsilon) = \{i \in \{1, \dots, N\} : c_i(x) \leq V_c(x) - \epsilon\}$$

$$\mathfrak{S}(x, \epsilon) = \{S \in \mathfrak{S} : \mathcal{N}(x, \epsilon) \subseteq S \text{ and } S \subseteq \mathcal{N}(x, \epsilon) \cup \mathcal{C}(x, \epsilon)\}$$

- Here: $\mathfrak{S}(x_*, \epsilon) = \{(1, 2, 4), (1, 2, 5)\}$

ϵ -critical scenario selections



$$\mathcal{C}(x, \epsilon) = \{i \in \{1, \dots, N\} : |c_i(x) - V_c(x)| \leq \epsilon\}$$

$$\mathcal{N}(x, \epsilon) = \{i \in \{1, \dots, N\} : c_i(x) \leq V_c(x) - \epsilon\}$$

$$\mathfrak{S}(x, \epsilon) = \{S \in \mathfrak{S} : \mathcal{N}(x, \epsilon) \subseteq S \text{ and } S \subseteq \mathcal{N}(x, \epsilon) \cup \mathcal{C}(x, \epsilon)\}$$

- Here: $\mathfrak{S}(x_k, \epsilon) = \{(1, 2, 4), (1, 2, 5)\}$

Subproblem with ϵ -critical scenario selections

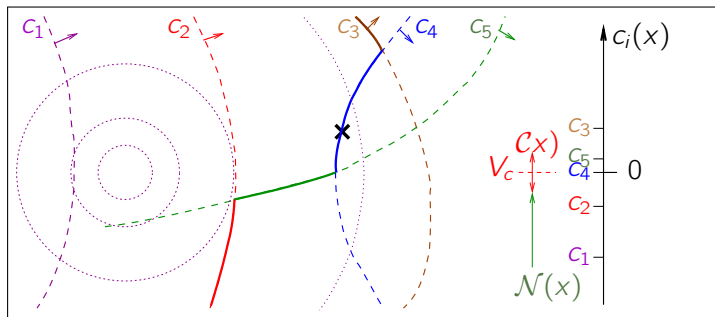
$$\begin{aligned} \min_{d \in \mathbb{R}^n} \quad & f_k + g_k^T d + \frac{1}{2} d^T H_k d + \rho \min_{\mathcal{S} \in \mathfrak{S}(x_k, \epsilon)} \| [c_{\mathcal{S}, k} + J_{\mathcal{S}, k} d]_+ \|_1 \\ \text{s.t.} \quad & \|d\|_\infty \leq \delta_k \end{aligned}$$

- ▶ Computational effort can be tuned with ϵ
 - ▶ Small ϵ : Easier subproblems
 - ▶ Large ϵ : “Broader view” of feasible region

Theorem 1

Suppose $f \in \mathcal{C}^1$, $c(\cdot, \xi) \in \mathcal{C}^1$, $\{H_k\}$ bounded, and $\phi(x_k)$ bounded below. Then, every limit point x_ of $\{x_k\}$ is a stationary point of $\phi_{\mathcal{S}}$ for all $\mathcal{S} \in \mathfrak{S}(x_*)$.*

Joint chance constraints



- ▶ Joint chance constraints: $m > 1$
- ▶ Combined constraint satisfaction:

$$V_i(x) = \begin{cases} \|[c_i(x)]_+\|_1 & \text{if } c_i(x) \not\leq 0 \\ \max\{c_{ij}(x) : j = 1, \dots, m\} & \text{if } c_i(x) \leq 0 \end{cases}$$

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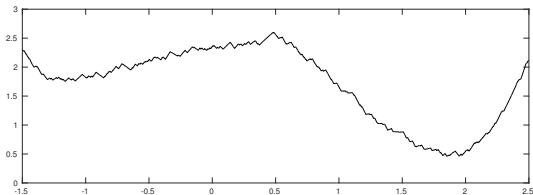
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Numerical experiments

- ▶ Matlab implementation with AMPL interface (suffixes)
- ▶ MIQPs solved by CPLEX
 - ▶ take incumbent after 5 min time limit (parallel with 8 threads)
- ▶ H_k : Hessian of Lagrangian for current scenario selection
- ▶ $\gamma \in (0, 1]$: include $\lceil \gamma N \rceil$ scenarios below and above $V_c(x_k)$ in \mathcal{C}

“Poor” local minimizers

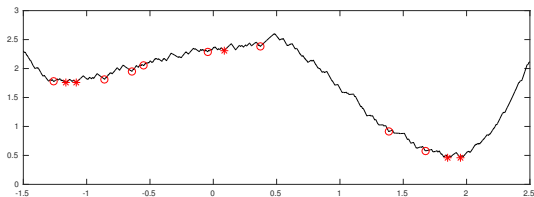


$$\min_{x, z}$$

$$\text{s.t. } \mathbb{P}[0.25x^4 - \frac{1}{3}x^3 - x^2 + 0.2x - 19.5 + \xi_1 x + \xi_1 \xi_0 \leq z] \geq 0.95$$

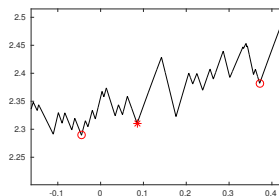
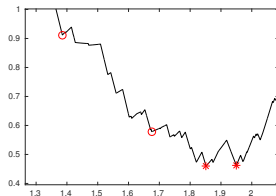
- ▶ ξ_0 from $U[-12, 12]$ and ξ_1 from $U[-3, 3]$
- ▶ $N = 5,000$

“Poor” local minimizers



		γ	0.001	0.002	0.005	0.010	0.050	0.100	0.200	1.000
x_0	N		11	21	51	101	501	751	1251	5000
-1.5			1.776	1.776	1.754	1.754	1.754	1.754	1.754	1.754
-1.0			1.754	1.760	1.754	1.754	1.754	1.754	1.754	1.754
-0.5			2.049	1.951	1.814	1.754	1.754	1.754	1.754	1.754
0.0			2.289	2.289	1.814	1.754	1.754	1.754	1.754	1.754
0.5			2.382	2.382	0.460	0.460	0.460	0.460	0.460	0.460
1.0			0.912	0.460	0.460	0.460	0.463	0.463	0.463	0.463
1.5			0.579	0.460	0.460	0.460	0.460	0.460	0.460	0.460
2.0			0.463	0.463	0.460	0.460	0.463	0.463	0.463	0.463
x^{bad}			2.310	2.289	1.814	1.754	1.754	1.754	1.754	1.754

“Poor” local minimizers



		γ								
		0.001	0.002	0.005	0.010	0.050	0.100	0.200	1.000	
x_0	N	11	21	51	101	501	751	1251	5000	
-1.5		1.776	1.776	1.754	1.754	1.754	1.754	1.754	1.754	
-1.0		1.754	1.760	1.754	1.754	1.754	1.754	1.754	1.754	
-0.5		2.049	1.951	1.814	1.754	1.754	1.754	1.754	1.754	
0.0		2.289	2.289	1.814	1.754	1.754	1.754	1.754	1.754	
0.5		2.382	2.382	0.460	0.460	0.460	0.460	0.460	0.460	
1.0		0.912	0.460	0.460	0.460	0.463	0.463	0.463	0.463	
1.5		0.579	0.460	0.460	0.460	0.460	0.460	0.460	0.460	
2.0		0.463	0.463	0.460	0.460	0.463	0.463	0.463	0.463	
x^{bad}		2.310	2.289	1.814	1.754	1.754	1.754	1.754	1.754	

Cash flow problem [Dentcheva et al. 03]

- ▶ Invest capital over time horizon into different investment options
- ▶ Make sure we can pay random liabilities (with 95% prob.)
- ▶ Maximize final cash
- ▶ Interest rates depend on amount of investment
 - ▶ This makes the problem nonconvex
- ▶ 5 random instances per size

Results

N	γ	instances solved	iter	$ \mathcal{C}_k $	changes in \mathcal{C}_k	% improve over robust	time in secs
500	0.001	5	9.84	9.40	6.14	5.7223 (5)	25.95
500	0.005	5	8.19	11.84	8.68	5.9602 (5)	37.45
500	0.050	5	7.04	51.00	13.06	6.0349 (5)	52.96
500	1.000	5	7.04	500.00	15.52	6.0349 (5)	260.08
1000	0.001	5	15.98	10.90	15.15	7.4040 (5)	117.13
1000	0.005	5	12.29	15.22	23.72	7.7296 (5)	147.30
1000	0.050	5	8.12	101.00	34.29	7.9134 (5)	313.17
1000	1.000	5	7.43	1000.00	57.14	7.9134 (5)	819.24
2000	0.001	5	21.94	13.84	43.67	9.7347 (4)	652.19
2000	0.005	4	12.85	23.01	56.61	9.9029 (4)	513.20
2000	0.050	4	9.90	201.00	88.99	10.1009 (4)	1099.41
2000	1.000	5	14.33	2000.00	129.18	10.0924 (4)	3519.53

- ▶ 96 vars, 20 chance constr, 150 non-chance constr
- ▶ Size for $N = 2000$ and $\gamma = 0.20$:
 - ▶ MIQP has about 40,000 constraints and 500 binary variables

Outline

Problem Statements

Penalty Function

Algorithm

Experiments

Conclusion

Conclusions

- ▶ SQP-type trust-region algorithm
 - ▶ Exact penalty function for chance-constrained NLP
- ▶ Computational effort of subproblem can be tuned
 - ▶ Small ϵ : Reduce solution time
 - ▶ Large ϵ : Escape spurious local solutions
- ▶ Potential improvements
 - ▶ Use efficient branch-and-cut subproblem (e.g., [Luedtke 12], ...)
 - ▶ Adaptive choices of ρ and/or ϵ or γ
 - ▶ Adaptive sample size
 - ▶ At beginning, consider small subset of scenarios and choose large ϵ
 - ▶ Refine SAA discretization with proximity to solution
 - ▶ Heuristic solutions of subproblem
 - ▶ Avoid solution of mixed-integer problem
 - ▶ E.g., progress along successively improving scenario selections
 - ▶ Ignore “very satisfied” constraints in subproblem
 - ▶ Do not include scenarios with $c_i(x) \ll 0$