Characterizing the Worst-Case Performance of Algorithms for Nonconvex Optimization

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joint work with

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Outline

Motivation

Contemporary Analyses

Partitioning the Search Space

Behavior of Common Methods

Summary & Perspectives
Outline

- Motivation
- Contemporary Analyses
- Partitioning the Search Space
- Behavior of Common Methods
- Summary & Perspectives
Consider the problem to minimize an objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\min_{x \in \mathbb{R}^n} f(x).$$

Various iterative algorithms have been proposed of the form

$$x_{k+1} \leftarrow x_k + s_k \quad \text{for all } k \in \mathbb{N},$$

where $\{x_k\}$ is the iterate sequence and $\{s_k\}$ is the step sequence.

For the purposes of this talk on nonconvex optimization . . .

- not going to do *global* optimization;
- focus on deterministic methods, though ideas could be extended to stochastic
History

Nonlinear optimization algorithm design has had parallel developments:

- Convexity
  - Rockafellar
  - Fenchel
  - Nemirovski
  - Nesterov
- Subgradient inequality
- Convergence, complexity guarantees

- Smoothness
  - Powell
  - Fletcher
  - Goldfarb
  - Nocedal
- Sufficient decrease
- Convergence, fast local convergence

Worlds are finally colliding!
Worst-case complexity for convex optimization

**Worst-case complexity:** Upper limit on the resources an algorithm will require to (approximately) solve a given problem.
Worst-case complexity for convex optimization

**Worst-case complexity:** Upper limit on the resources an algorithm will require to (approximately) solve a given problem

**... for convex optimization:** Bound on the number of iterations (or function or derivative evaluations) until

\[ \|x_k - x^*\| \leq \epsilon_x \]

or

\[ f(x_k) - f(x^*) \leq \epsilon_f, \]

where \( x^* \) is some global minimizer of \( f \).
Worst-case complexity for convex optimization

**Worst-case complexity:** Upper limit on the resources an algorithm will require to (approximately) solve a given problem

...for convex optimization: Bound on the number of iterations (or function or derivative evaluations) until

$$\|x_k - x_*\| \leq \epsilon_x$$

or

$$f(x_k) - f(x_*) \leq \epsilon_f,$$

where $x_*$ is some global minimizer of $f$.

**Fact(?):** Convex setting: better complexity *often* implies better performance.

(Really, need to consider work complexity, conditioning, structure, etc.)
...for nonconvex optimization: Here is how we do it now:

Since one generally cannot guarantee that \( \{x_k\} \) converges to a minimizer, one asks for an upper bound on the number of iterations until

\[
\|\nabla f(x_k)\| \leq \epsilon_g \quad \text{(first-order stationarity)}
\]

and maybe also \( \lambda(\nabla^2 f(x_k)) \geq -\epsilon_H \) (second-order stationarity)
Worst-case complexity for nonconvex optimization

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Since one generally cannot guarantee that \( \{x_k\} \) converges to a minimizer, one asks for an upper bound on the number of iterations until

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\| \nabla f(x_k) \| \leq \epsilon_g \quad \text{(first-order stationarity)}
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and maybe also \( \lambda(\nabla^2 f(x_k)) \geq -\epsilon_H \quad \text{(second-order stationarity)} \)

For example, it is known that for first-order stationarity we have the bounds...

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>until ( | \nabla f(x_k) |_2 \leq \epsilon_g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient descent</td>
<td>( O(\epsilon_g^{-2}) )</td>
</tr>
<tr>
<td>Newton / trust region</td>
<td>( O(\epsilon_g^{-2}) )</td>
</tr>
<tr>
<td>Cubic regularization</td>
<td>( O(\epsilon_g^{-3/2}) )</td>
</tr>
</tbody>
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Self-examination

But…

- Is this the best way to *characterize* our algorithms?
- Is this the best way to *represent* our algorithms?
Self-examination

But...

▶ Is this the best way to \textit{characterize} our algorithms?
▶ Is this the best way to \textit{represent} our algorithms?

People listen! Cubic regularization...

▶ Griewank (1981)
▶ Nesterov & Polyak (2006)
▶ Weiser, Deuflhard, Erdmann (2007)
▶ Cartis, Gould, Toint (2011), the \textit{ARC} method

...is a framework to which researchers have been attracted...

▶ Agarwal, Allen-Zhu, Bullins, Hazan, Ma (2017)
▶ Carmon, Duchi (2017)
▶ Kohler, Lucchi (2017)
▶ Peng, Roosta-Khorasan, Mahoney (2017)

However, there remains a large gap between theory and practice!

(Trust region methods arguably perform better in general.)
Symmetric low-rank matrix factorization problem:

\[
\min_{X \in \mathbb{R}^{d \times r}} \frac{1}{2} \| X X^T - M \|_F^2,
\]

where \( M \in \mathbb{R}^{d \times d} \) with \( \text{rank}(M) = r \).

- Nonconvex, but...
- Global minimum value is known (it’s zero)
- All local minima are global minima

Jin, Ge, Netrapalli, Kakade, Jordan (2017)
Example: Dictionary learning

Learning a representation of input data in the form of linear combinations of some (unknown) basic elements, called *atoms*, which compose a *dictionary*:

\[
\min_{X \in \mathcal{X}, Y \in \mathbb{R}^{n \times n}} \|Z - XY\|^2 + \phi(Y)
\]

s.t. \( \mathcal{X} := \{X \in \mathbb{R}^{d \times n} : \|X_i\|_2 \leq 1 \text{ for all } i \in \{1, \ldots, n\}\} \),

where \( Z \in \mathbb{R}^{d \times n} \) is a given input.

Nonconvex, but, under some conditions, all saddle points can be “escaped”.
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where \(Z \in \mathbb{R}^{d \times n}\) is a given input.

Nonconvex, but, under some conditions, all saddle points can be “escaped”.

Sun, Qu, Wright (2016)
Other examples

- Phase retrieval
- Orthogonal tensor decomposition
- Deep linear learning
- ...
Pedagogical example

But if we’re talking about nonconvex optimization, we also could have...

What *real* problem exhibits this behavior? (I don’t know!)

More on this example later...
Purpose of this talk

Our goal: A *complementary* approach to characterize algorithms.

- global convergence
- worst-case complexity, contemporary type + our approach
- local convergence rate
Purpose of this talk

Our goal: A *complementary* approach to characterize algorithms.

- global convergence
- worst-case complexity, contemporary type + *our approach*
- local convergence rate

We’re admitting: Our approach *does not always* give the complete picture.

But we believe it *is* useful.
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Summary & Perspectives
Simple setting

Suppose the gradient \( g := \nabla f \) is Lipschitz continuous with constant \( L > 0 \).

Consider the iteration (with \( g_k := \nabla f(x_k) \))

\[
x_{k+1} \leftarrow x_k - \frac{1}{L} g_k \quad \text{for all } k \in \mathbb{N}.
\]

A contemporary complexity analysis considers the set

\[
G(\epsilon_g) := \{ x \in \mathbb{R}^n : \|g(x)\|_2 \leq \epsilon_g \}
\]

and aims to find an upper bound on the cardinality of

\[
K_g(\epsilon_g) := \{ k \in \mathbb{N} : x_k \not\in G(\epsilon_g) \}.
\]
Upper bound on $|\mathcal{K}_g(\epsilon_g)|$

Using $s_k = -\frac{1}{L}g_k$ and the upper bound

$$f_{k+1} \leq f_k + g_k^T s_k + \frac{1}{2} L \|s_k\|^2_2,$$

one finds with $f_{\text{inf}} := \inf_{x \in \mathbb{R}^n} f(x)$ that

$$f_k - f_{k+1} \geq \frac{1}{2L} \|g_k\|^2_2$$

$$\implies (f_0 - f_{\text{inf}}) \geq \frac{1}{2L} |\mathcal{K}_g(\epsilon_g)| \epsilon_g^2$$

$$\implies |\mathcal{K}_g(\epsilon_g)| \leq 2L(f_0 - f_{\text{inf}})\epsilon_g^{-2}.$$
But what if \( f \) is “nice”?

e.g., satisfying the Polyak-Łojasiewicz condition for \( c \in (0, \infty) \), i.e.,

\[
f(x) - f_{\inf} \leq \frac{1}{2c} \|g(x)\|_2^2 \quad \text{for all } x \in \mathbb{R}^n.
\]

Now consider the set

\[
\mathcal{F}(\epsilon_f) := \{ x \in \mathbb{R}^n : f(x) - f_{\inf} \leq \epsilon_f \}
\]

and consider an upper bound on the cardinality of

\[
\mathcal{K}_f(\epsilon_f) := \{ k \in \mathbb{N} : x_k \not\in \mathcal{F}(\epsilon_f) \}.
\]
Upper bound on $|\mathcal{K}_f(\epsilon_f)|$

Using $s_k = -\frac{1}{L}g_k$ and the upper bound

$$f_{k+1} \leq f_k + g_k^T s_k + \frac{1}{2}L\|s_k\|_2^2,$$

one finds that

$$f_k - f_{k+1} \geq \frac{1}{2L}\|g_k\|_2^2 \geq \frac{c}{L}(f_k - f_{\text{inf}})$$

$$\implies (1 - \frac{c}{L})(f_k - f_{\text{inf}}) \geq f_{k+1} - f_{\text{inf}}$$

$$\implies (1 - \frac{c}{L})^k(f_0 - f_{\text{inf}}) \geq f_k - f_{\text{inf}}$$

$$\implies |\mathcal{K}_f(\epsilon_f)| \leq \log \left( \frac{f_0 - f_{\text{inf}}}{\epsilon_f} \right) \left( \log \left( \frac{L}{L-c} \right) \right)^{-1}.$$
For the first step...

In the “general nonconvex” analysis...

...the expected decrease for the first step is much more pessimistic:

\[ f_0 - f_1 \geq \frac{1}{2L} \epsilon_g^2 \]

PL condition: \( (1 - \frac{c}{L})(f_0 - f_{\text{inf}}) \geq f_1 - f_{\text{inf}} \)

...and it remains more pessimistic throughout!
Upper bounds on $|\mathcal{K}_f(\epsilon_f)|$ versus $|\mathcal{K}_g(\epsilon_g)|$

Let $f(x) = \frac{1}{2}x^2$, meaning that $g(x) = x$.

- Let $\epsilon_f = \frac{1}{2}\epsilon_g^2$, meaning that $\mathcal{F}(\epsilon_f) = \mathcal{G}(\epsilon_g)$.
- Let $x_0 = 10$, $c = 1$, and $L = 2$. (Similar pictures for any $L > 1$.)
Upper bounds on $|\mathcal{K}_f(\epsilon_f)|$ versus $|\{k \in \mathbb{N} : \frac{1}{2}\|g_k\|_2^2 > \epsilon_g\}|$

Let $f(x) = \frac{1}{2}x^2$, meaning that $\frac{1}{2}g(x)^2 = \frac{1}{2}x^2$.

- Let $\epsilon_f = \epsilon_g$, meaning that $\mathcal{F}(\epsilon_f) = \mathcal{G}(\epsilon_g)$.

- Let $x_0 = 10$, $c = 1$, and $L = 2$. (Similar pictures for any $L > 1$.)
Bad worst-case!

Worst-case complexity bounds in the general nonconvex case are very pessimistic.

- The analysis immediately admits a large gap when the function is nice.
- The “essentially tight” examples for the worst-case bounds are... weird.\(^1\)

\(^1\)Cartis, Gould, Toint (2010)
Outline

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**Contemporary Analyses**

**Partitioning the Search Space**

**Behavior of Common Methods**

**Summary & Perspectives**
We want a characterization strategy that

- attempts to capture behavior in *actual practice*
- i.e., is not “bogged down” by pedagogical examples
- can be applied consistently across different classes of functions
- shows more than just the worst of the worst case
Motivation

We want a characterization strategy that

- attempts to capture behavior in *actual practice*
- i.e., is not “bogged down” by pedagogical examples
- can be applied consistently across different classes of functions
- shows more than just the worst of the worst case

Our idea is to

- partition the search space (dependent on $f$ and $x_0$)
- analyze how an algorithm behaves over different regions
- characterize an algorithm’s behavior *by region*

For some functions, there will be holes, but for some of interest there are none!
Think about an arbitrary point in the search space, i.e.,

\[ \mathcal{L} := \{ x \in \mathbb{R}^n : f(x) \leq f(x_0) \}. \]

- If \( \| g(x) \|_2 \gg 0 \), then “a lot” of progress can be made.
- If \( \lambda(\nabla^2 f(x)) \ll 0 \), then “a lot” of progress can also be made.
Assumption

Assumption 1

- \( f \) is \( \bar{p} \)-times continuously differentiable
- \( f \) is bounded below by \( f_{\text{inf}} := \inf_{x \in \mathbb{R}^n} f(x) \)
- For all \( p \in \{1, \ldots, \bar{p}\} \), there exists \( L_p \in (0, \infty) \) such that

\[
f(x + s) \leq f(x) + \sum_{j=1}^{\bar{p}} \frac{1}{j!} \nabla^j f(x)[s]^j + \frac{L_p}{p + 1} \|s\|^{p+1}
\]

\[\text{for } t_p(x,s)\]
**Definition 2**

For each $p \in \{1, \ldots, \bar{p}\}$, define the function

$$m_p(x, s) = \frac{1}{p!} \nabla^p f(x)[s]^p + \frac{r_p}{p + 1} \|s\|^{p+1}_2.$$  

Letting $s_{m_p}(x) := \arg \min_{s \in \mathbb{R}^n} m_p(x, s)$, the reduction in the $p$th-order term from $x$ is

$$\Delta m_p(x) = m_p(x, 0) - m_p(x, s_{m_p}(x)) \geq 0.$$  

*Exact definition of $r_p$ is not complicated, but we’ll skip it here*
1st-order and 2nd-order term reductions

Theorem 3

For \( \bar{p} \geq 2 \), the following hold:

\[
\Delta m_1(x) = \frac{1}{2r_1} \| \nabla f(x) \|_2^2
\]

and

\[
\Delta m_2(x) = \frac{1}{6r_2^2} \max \{-\lambda(\nabla^2 f(x_k)), 0\}^3.
\]
We propose to partition the search space, given \((\kappa, f_{\text{ref}}) \in (0, 1) \times [f_{\text{inf}}, f(x_0)]\), into
\[
R_1 := \{x \in \mathcal{L} : \Delta m_1(x) \geq \kappa(f(x) - f_{\text{ref}})\},
\]
\[
R_p := \{x \in \mathcal{L} : \Delta m_p(x) \geq \kappa(f(x) - f_{\text{ref}})\} \setminus \left( \bigcup_{j=1}^{p-1} R_j \right) \quad \text{for all } p \in \{2, \ldots, \bar{p}\},
\]
and
\[
\bar{R} := \mathcal{L} \setminus \left( \bigcup_{j=1}^{\bar{p}} R_j \right).
\]

*We don’t need \(f_{\text{ref}} = f_{\text{inf}}\), but, for simplicity, think of it that way here.
Illustration

\((\bar{p} = 2)\) \quad \mathcal{R}_1: \text{black} \quad \mathcal{R}_2: \text{gray} \quad \mathcal{R}: \text{white}
Functions satisfying Polyak-Łojasiewicz

Theorem 4

A continuously differentiable $f$ with a Lipschitz continuous gradient satisfies the Polyak-Łojasiewicz condition if and only if $R_1 = L$ for any $x_0 \in \mathbb{R}^n$.

Hence, if we prove something about the behavior of an algorithm over $R_1$, then

- we know how it behaves if $f$ satisfies PL and
- we know how it behaves at any point satisfying the PL inequality.
Functions satisfying a strict-saddle-type property

Theorem 5

If $f$ is twice-continuously differentiable with Lipschitz continuous gradient and Hessian functions such that, at all $x \in \mathcal{L}$ and for some $\zeta \in (0, \infty)$, one has

$$\max\{\|\nabla f(x)\|^2_2, -\lambda (\nabla^2 f(x))^3\} \geq \zeta (f(x) - f_{inf}),$$

then $\mathcal{R}_1 \cup \mathcal{R}_2 = \mathcal{L}$. 
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Motivation Contemporary Analyses Partitioning Common Methods Summary

Linearly convergent behavior over $\mathcal{R}_p$

Let $s_{w_p}(x)$ be a minimum norm global minimizer of the regularized Taylor model

$$w_p(x, s) = t_p(x, s) + \frac{l_p}{p+1} \|s\|_2^{p+1}$$

**Theorem 6**

If $\{x_k\}$ is generated by the iteration

$$x_{k+1} \leftarrow x_k + s_{w_p}(x),$$

then, with $\epsilon_f \in (0, f(x_0) - f_{ref})$, the number of iterations in

$$\mathcal{R}_p \cap \{x \in \mathbb{R}^n : f(x) - f_{ref} \geq \epsilon_f\}$$

is bounded above by

$$\left\lceil \log \left( \frac{f(x_0) - f_{ref}}{\epsilon_f} \right) \left( \log \left( \frac{1}{1 - \kappa} \right) \right)^{-1} \right\rceil = \mathcal{O} \left( \log \left( \frac{f(x_0) - f_{ref}}{\epsilon_f} \right) \right)$$
Regularized gradient and Newton methods

▶ Regularized gradient method: Computes $s_k$ by solving

$$
\min_{s \in \mathbb{R}^n} f(x_k) + \nabla f(x_k)^T s + \frac{l_1}{2} \|s\|^2 \quad \implies \quad s_k = -\frac{1}{l_1} \nabla f(x_k)
$$

▶ Regularized Newton method: Computes $s_k$ by solving

$$
\min_{s \in \mathbb{R}^n} f(x_k) + \nabla f(x_k)^T s + \frac{1}{2} s^T \nabla^2 f(x_k) s + \frac{l_2}{3} \|s\|^3,
$$

also known as cubic regularization (mentioned earlier)
Let RG and RN represent regularized gradient and Newton, respectively.

Theorem 7

With $p \geq 2$, let

$$K_1(\epsilon_g) := \{ k \in \mathbb{N} : \| \nabla f(x_k) \|_2 > \epsilon_g \}$$

and $K_2(\epsilon_H) := \{ k \in \mathbb{N} : \lambda(\nabla^2 f(x_k)) < -\epsilon_H \}$.

Then, the cardinalities of $K_1(\epsilon_g)$ and $K_2(\epsilon_H)$ are of the order...

| Algorithm | $|K_1(\epsilon_g)|$ | $|K_2(\epsilon_H)|$ |
|-----------|------------------|------------------|
| RG        | $O \left( \frac{l_1(f(x_0) - f_{inf})}{\epsilon_g^2} \right)$ | $\infty$ |
| RN        | $O \left( \frac{l_2^{1/2}(f(x_0) - f_{inf})}{\epsilon_g^{3/2}} \right)$ | $O \left( \frac{l_2^2(f(x_0) - f_{inf})}{\epsilon_H^3} \right)$ |
Theorem 8

The numbers of iterations in $\mathcal{R}_1$ and $\mathcal{R}_2$ with $f_{\text{ref}} = f_{\text{inf}}$ are of the order...

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<td>$RG$</td>
<td>$O \left( \log \left( \frac{f(x_0) - f_{\text{inf}}}{\epsilon_f} \right) \right)$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$RN$</td>
<td>$O \left( \frac{l_2^2(f(x_0) - f_{\text{inf}})}{r_1^3} \right) + O \left( \log \left( \frac{f(x_0) - f_{\text{inf}}}{\epsilon_f} \right) \right)$</td>
<td>$O \left( \log \left( \frac{f(x_0) - f_{\text{inf}}}{\epsilon_f} \right) \right)$</td>
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There is an initial phase, as seen in Nesterov & Polyak (2006)
Characterization: Our approach

Theorem 8

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There is an initial phase, as seen in Nesterov & Polyak (2006)

A $\infty$ can appear, but one could consider probabilistic bounds, too
Trust region method: Gradient-dependent radii

\[
\min_{s \in \mathbb{R}^n} f(x_k) + \nabla f(x_k)^T s + \frac{1}{2} s^T \nabla^2 f(x_k) s \quad \text{s.t.} \quad \|s\|_2 \leq \delta_k
\]

- Set \( \delta_k \leftarrow \nu_k \|\nabla f(x_k)\|_2 \)
- Initialize \( \nu_0 \in [\nu, \bar{\nu}] \)
- For some \((\eta, \beta) \in (0, 1) \times (0, 1), \) if

\[
\rho_k = \frac{f(x_k) - f(x_k + s_k)}{t_2(x_k, 0) - t_2(x_k, s_k)} \geq \eta,
\]

then \( x_{k+1} \leftarrow x_k + s_k \) and \( \nu_{k+1} \in [\nu, \bar{\nu}] \); else, \( x_{k+1} \leftarrow x_k \) and \( \nu_{k+1} \leftarrow \beta \nu_k \).
Trust region method: Gradient-dependent radii

\[
\min_{s \in \mathbb{R}^n} \ f(x_k) + \nabla f(x_k)^T s + \frac{1}{2} s^T \nabla^2 f(x_k) s \quad \text{s.t.} \quad \|s\|_2 \leq \delta_k
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**Theorem 9**

# of iterations in \( \mathcal{R}_1 \) is at most \( \mathcal{O} \left( \chi \log \left( \frac{f(x_0) - f_{\text{ref}}}{\epsilon_f} \right) \right) \). For \( \mathcal{R}_2 \), no guarantee.
Trust region method: Gradient- and Hessian-dependent radii

\[
\min_{s \in \mathbb{R}^n} f(x_k) + \nabla f(x_k)^T s + \frac{1}{2} s^T \nabla^2 f(x_k) s \quad \text{s.t.} \quad \|s\|_2 \leq \delta_k
\]

- Set

\[
\delta_k \leftarrow \nu_k \begin{cases} 
\|\nabla f(x_k)\|_2 & \|\nabla f(x_k)\|_2^2 \geq |\lambda(\nabla^2 f(x_k))|^3 \\
|\lambda(\nabla^2 f(x_k))| & \text{otherwise}
\end{cases}
\]

- Initialize \(\nu_0 \in [\nu, \bar{\nu}]\)

- For some \((\eta, \beta) \in (0, 1) \times (0, 1)\), if

\[
\rho_k = \frac{f(x_k) - f(x_k + s_k)}{t_2(x_k, 0) - t_2(x_k, s_k)} \geq \eta,
\]

then \(x_{k+1} \leftarrow x_k + s_k\) and \(\nu_{k+1} \in [\nu, \bar{\nu}]\); else, \(x_{k+1} \leftarrow x_k\) and \(\nu_{k+1} \leftarrow \beta \nu_k\).
Trust region method: Gradient- and Hessian-dependent radii

\[
\min_{s \in \mathbb{R}^n} f(x_k) + \nabla f(x_k)^T s + \frac{1}{2} s^T \nabla^2 f(x_k) s \quad \text{s.t.} \quad \|s\|_2 \leq \delta_k
\]

- Set
  \[
  \delta_k \leftarrow \nu_k \begin{cases} 
  \|\nabla f(x_k)\|_2 & \|\nabla f(x_k)\|_2^2 \geq |\lambda(\nabla^2 f(x_k))|^3 \\
  |\lambda(\nabla^2 f(x_k))| & \text{otherwise}
  \end{cases}
  \]

- Initialize \( \nu_0 \in [\nu, \bar{\nu}] \)

- For some \((\eta, \beta) \in (0, 1) \times (0, 1)\), if
  \[
  \rho_k = \frac{f(x_k) - f(x_k + s_k)}{t_2(x_k, 0) - t_2(x_k, s_k)} \geq \eta,
  \]
  then \( x_{k+1} \leftarrow x_k + s_k \) and \( \nu_{k+1} \in [\nu, \bar{\nu}] \); else, \( x_{k+1} \leftarrow x_k \) and \( \nu_{k+1} \leftarrow \beta \nu_k \).

**Theorem 10**

\# of iterations in \( R_1 \) is at most \( \mathcal{O}\left(\chi \log\left(\frac{f(x_0) - f_{\text{ref}}}{\epsilon_f}\right)\right) \).

\# of iterations in \( R_2 \) is at most \( \mathcal{O}\left(\chi_2 \log\left(\frac{f(x_0) - f_{\text{ref}}}{\epsilon_f}\right)\right) \).
Trust region method: Always good?

What about the classical update?

$$\delta_{k+1} \leftarrow \begin{cases} \geq \delta_k & \text{if } \rho_k \geq \eta \\ < \delta_k & \text{otherwise.} \end{cases}$$

Two challenges:

- Proving a uniform upper bound on number of consecutive rejected steps
- Proving that accepted steps yield sufficient decrease in $R_1$ and $R_2$
Outline

Motivation

Contemporary Analyses

Partitioning the Search Space

Behavior of Common Methods

Summary & Perspectives
Our goal: A *complementary* approach to characterize algorithms.

- global convergence
- worst-case complexity, contemporary type + our approach
- local convergence rate

Our idea is to

- partition the search space (dependent on $f$ and $x_0$)
- analyze how an algorithm behaves over different regions
- characterize an algorithm’s behavior *by region*

For some functions, there are holes, but for others the characterization is complete.