Algorithms for Deterministically Constrained Stochastic Optimization

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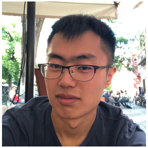
Collaborators and references






- A. S. Berahas, F. E. Curtis, D. P. Robinson, and B. Zhou, "Sequential Quadratic Optimization for Nonlinear Equality Constrained Stochastic Optimization," *SIAM Journal on Optimization*, 31(2):1352–1379, 2021.
- A. S. Berahas, F. E. Curtis, M. J. O'Neill, and D. P. Robinson, "A Stochastic Sequential Quadratic Optimization Algorithm for Nonlinear Equality Constrained Optimization with Rank-Deficient Jacobians," https://arxiv.org/abs/2106.13015.
- F. E. Curtis, D. P. Robinson, and B. Zhou, "Inexact Sequential Quadratic Optimization for Minimizing a Stochastic Objective Subject to Deterministic Nonlinear Equality Constraints," https://arxiv.org/abs/2107.03512.
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Outline

Motivation

SG and SQP

Adaptive (Deterministic) SQP

Stochastic SQP

Worst-Case Iteration Complexity

Extensions

Conclusion

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Constrained optimization (deterministic)

Consider

$$\boxed{\begin{aligned} \min_{x\in\mathbb{R}^n}\ & f(x) \\ \text{s.t. } & c_{\mathcal{E}}(x) = 0 \\ & c_{\mathcal{I}}(x) \leq 0 \end{aligned}}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $c_{\mathcal{E}} : \mathbb{R}^n \to \mathbb{R}^{m_{\mathcal{E}}}$, and $c_{\mathcal{I}} : \mathbb{R}^n \to \mathbb{R}^{m_{\mathcal{I}}}$ are smooth

- Physics-constrained, resource-constrained, etc.
- Long history of algorithms (penalty, SQP, interior-point, etc.)
- Comprehensive theory (even with lack of constraint qualifications)
- Effective software (Ipopt, Knitro, LOQO, etc.)

Constrained optimization (stochastic constraints)

Consider

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & f(x) \\ \text{s.t.} \ & c_{\mathcal{E}}(x) = 0 \\ & c_{\mathcal{I}}(x, \omega) \lesssim 0 \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $c_{\mathcal{E}} : \mathbb{R}^n \to \mathbb{R}^{m_{\mathcal{E}}}$, and $c_{\mathcal{I}} : \mathbb{R}^n \times \Omega \to \mathbb{R}^{m_{\mathcal{I}}}$

- Various modeling paradigms:
- . . . stochastic optimization
- . . . (distributionally) robust optimization
- . . . chance-constrained optimization

Constrained optimization (stochastic objective)

Consider

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & f(x) \equiv \mathbb{E}[F(x,\omega)] \\ \text{s.t. } & c_{\mathcal{E}}(x) = 0 \\ & c_{\mathcal{I}}(x) \leq 0 \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}$, $F : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$, $c_{\mathcal{E}} : \mathbb{R}^n \rightarrow \mathbb{R}^{m_{\mathcal{E}}}$, and $c_{\mathcal{I}} : \mathbb{R}^n \rightarrow \mathbb{R}^{m_{\mathcal{I}}}$

- ω has probability space $(\Omega, \mathcal{F}, P)$
- $\mathbb{E}[\cdot]$ with respect to P
- Classical applications under uncertainty, constrained DNN training, etc.
- Besides cases involving a deterministic equivalent...
- ... very few algorithms so far (mostly penalty methods)

What kind of algorithm do we want?

Need to establish what we want/expect from an algorithm.

Note: We are interested in the fully stochastic regime.†

We assume:

- Feasible methods are not tractable
- ... so no projection methods, Frank-Wolfe, etc.
- "Two-phase" methods are not effective
- ... so should not search for feasibility, then optimize.
- Only enforce convergence in expectation.

Finally, want to use techniques that can generalize to diverse settings.

† Alternatively, see Na, Anitescu, Kolar (2021, 2022)

This talk

Consider *equality constrained* stochastic optimization:

$$\min_{x\in\mathbb{R}^n} f(x) \equiv \mathbb{E}[F(x,\omega)]$$
$$\text{s.t. } c_{\mathcal{E}}(x) = 0$$

- *Adaptive* SQP method for deterministic setting
- *Stochastic* SQP method for stochastic setting
- Convergence in expection (comparable to SG for unconstrained setting)
- Worst-case complexity on par with stochastic subgradient method
- Numerical experiments are very promising
- Various open questions!

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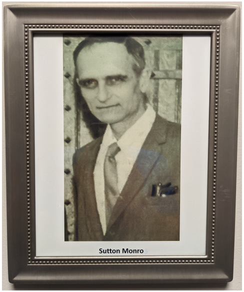
Stochastic gradient method (SG)

Invented by Herbert Robbins and Sutton Monro (1951)



Sutton Monro, former Lehigh faculty member

Stochastic gradient (*not* descent)

$$\boxed{\min_{x \in \mathbb{R}^n} \ f(x) \equiv \mathbb{E}[F(x,\omega)]}$$

where $g := \nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with constant L

Algorithm SG : Stochastic Gradient

choose an initial point $x_0 \in \mathbb{R}^n$ and step sizes $\{\alpha_k\} > 0$
for $k \in \{0, 1, 2, \dots\}$ **do**
  set $x_{k+1} \leftarrow x_k - \alpha_k \bar{g}_k$, where $\mathbb{E}_k[\bar{g}_k] = g_k$ and $\mathbb{E}_k[\|\bar{g}_k - g_k\|_2^2] \leq M$
end for

Not a descent method! ...but *eventual descent in expectation*:

$$\begin{aligned} f(x_{k+1}) - f(x_k) &\leq g_k^T(x_{k+1} - x_k) + \tfrac{1}{2}L\|x_{k+1} - x_k\|_2^2 \\ &= -\alpha_k g_k^T \bar{g}_k + \tfrac{1}{2}\alpha_k^2 L \|\bar{g}_k\|_2^2 \\ \implies \mathbb{E}_k[f(x_{k+1})] - f(x_k) &\leq -\alpha_k \|g_k\|_2^2 + \tfrac{1}{2}\alpha_k^2 L \mathbb{E}_k[\|\bar{g}_k\|_2^2]. \end{aligned}$$

Markovian: x_{k+1} depends only on x_k and random choice at iteration k.

SG theory

Theorem SG

Since $\mathbb{E}_k[\bar{g}_k] = g_k$ *and* $\mathbb{E}_k[\|\bar{g}_k - g_k\|_2^2] \leq M$ *for all* $k \in \mathbb{N}$*:*

$$\alpha_k = \frac{1}{L} \implies \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{k}\|g_j\|_2^2\right] \leq \mathcal{O}(M)$$

$$\alpha_k = \Theta\left(\frac{1}{k}\right) \implies \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^{k}\alpha_j\right)}\sum_{j=1}^{k}\alpha_j\|g_j\|_2^2\right] \to 0$$

$$\implies \liminf_{k\to\infty}\ \mathbb{E}[\|g_k\|_2^2] = 0$$

SG illustration

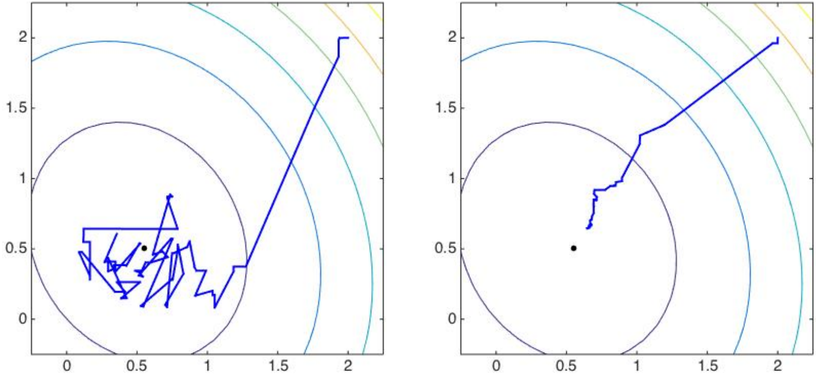


Figure: SG with fixed step size (left) vs. diminishing step sizes (right)

Sequential quadratic optimization (SQP)

Consider

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{s.t. } c(x) = 0$$

with $g \equiv \nabla f$, $J \equiv \nabla c$, and $H \succ 0$ (for simplicity), two viewpoints:

$$\begin{bmatrix} g(x) + J(x)^T y \\ c(x) \end{bmatrix} = 0$$

or

$$\min_{d\in\mathbb{R}^n} f(x) + g(x)^T d + \tfrac{1}{2} d^T H d \quad \text{s.t. } c(x) + J(x)d = 0$$

both leading to the same "Newton-SQP system":

$$\begin{bmatrix} H_k & J_k^T \\ J_k & 0 \end{bmatrix} \begin{bmatrix} d_k \\ y_k \end{bmatrix} = - \begin{bmatrix} g_k \\ c_k \end{bmatrix}$$

SQP illustration

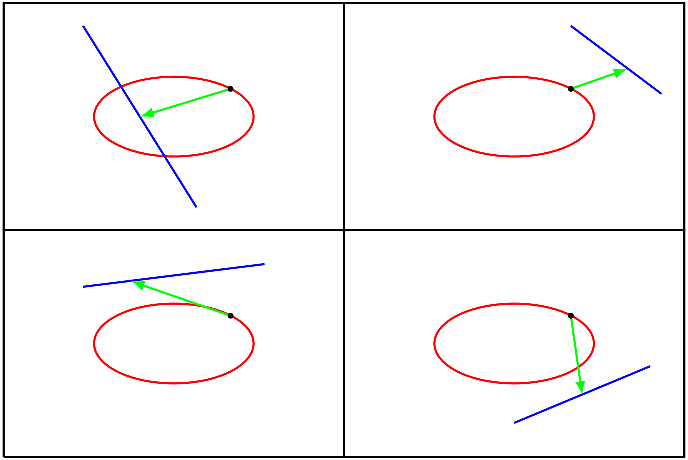


Figure: Illustrations of SQP subproblem solutions

SQP

- Algorithm guided by merit function, with adaptive parameter τ, defined by

$$\phi(x,\tau) = \tau f(x) + \|c(x)\|_1$$

a model of which is defined as

$$q(x,\tau,d) = \tau(f(x) + g(x)^T d + \tfrac{1}{2} d^T H d) + \|c(x) + J(x)d\|_1$$

- For a given $d \in \mathbb{R}^n$ satisfying $c(x) + J(x)d = 0$, the reduction in this model is

$$\Delta q(x,\tau,d) = -\tau(g(x)^T d + \tfrac{1}{2} d^T H d) + \|c(x)\|_1,$$

and it is easily shown that

$$\phi'(x,\tau,d) \leq -\Delta q(x,\tau,d)$$

SQP with backtracking line search

Algorithm SQP-B

choose $x_0 \in \mathbb{R}^n$, $\tau_{-1} \in \mathbb{R}_{>0}$, $\sigma \in (0,1)$, $\eta \in (0,1)$
for $k \in \{0,1,2,\dots\}$ **do**
  solve
$$\begin{bmatrix} H_k & J_k^T \\ J_k & 0 \end{bmatrix} \begin{bmatrix} d_k \\ y_k \end{bmatrix} = - \begin{bmatrix} g_k \\ c_k \end{bmatrix}$$
  set τ_k to ensure $\Delta q(x_k, \tau_k, d_k) \gg 0$, offered by
$$\tau_k \leq \frac{(1-\sigma)\|c_k\|_1}{g_k^T d_k + d_k^T H_k d_k} \quad \text{if} \quad g_k^T d_k + d_k^T H_k d_k > 0$$
  backtracking line search to ensure $x_{k+1} \leftarrow x_k + \alpha_k d_k$ yields
$$\phi(x_{k+1}, \tau_k) \leq \phi(x_k, \tau_k) - \eta \alpha_k \Delta q(x_k, \tau_k, d_k)$$
end for

Convergence theory

Assumption

- *f, c, g, and J bounded and Lipschitz*
- *singular values of J bounded below (i.e., the LICQ)*
- $u^T H_k u \geq \zeta \|u\|_2^2$ *for all* $u \in \text{Null}(J_k)$ *for all* $k \in \mathbb{N}$

Theorem SQP-B

- $\{\alpha_k\} \geq \alpha_{\min}$ *for some* $\alpha_{\min} > 0$
- $\{\tau_k\} \geq \tau_{\min}$ *for some* $\tau_{\min} > 0$
- $\Delta q(x_k, \tau_k, d_k) \to 0$ *implies*

$$\|d_k\|_2 \to 0, \quad \|c_k\|_2 \to 0, \quad \|g_k + J_k^T y_k\|_2 \to 0$$

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Toward stochastic SQP

- In a stochastic setting, line searches are (likely) intractable
- However, for ∇f and ∇c, may have Lipschitz constants (or estimates)
- Step #1: Design an adaptive SQP method with

  step sizes determined by Lipschitz constant estimates

- Step #2: Design a stochastic SQP method on this approach

Primary challenge: Nonsmoothness

In SQP-B, step size is chosen based on reducing the merit function.

The merit function is nonsmooth! An upper bound is

$$\begin{aligned}&\phi(x_k + \alpha_k d_k, \tau_k) - \phi(x_k, \tau_k)\\ \leq\ &\alpha_k \tau_k g_k^T d_k + |1 - \alpha_k| \|c_k\|_1 - \|c_k\|_1 + \tfrac{1}{2}(\tau_k L_k + \Gamma_k)\alpha_k^2 \|d_k\|_2^2\end{aligned}$$

where L_k and Γ_k are Lipschitz constant estimates for f and $\|c\|_1$ at x_k

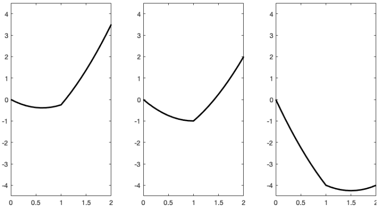

Figure: Three cases for upper bound of ϕ

Idea: Choose α_k to ensure sufficient decrease using this bound

SQP with adaptive step sizes

Algorithm SQP-A

1: choose $x_0 \in \mathbb{R}^n$, $\tau_{-1} \in \mathbb{R}_{>0}$, $\sigma \in (0,1)$, $\eta \in (0,1)$
2: **for** $k \in \{0,1,2,\dots\}$ **do**
3: solve
$$\begin{bmatrix} H_k & J_k^T \\ J_k & 0 \end{bmatrix} \begin{bmatrix} d_k \\ y_k \end{bmatrix} = - \begin{bmatrix} g_k \\ c_k \end{bmatrix}$$
4: set τ_k to ensure $\Delta q(x_k, \tau_k, d_k) \gg 0$, offered by
$$\tau_k \leq \frac{(1-\sigma)\|c_k\|_1}{g_k^T d_k + d_k^T H_k d_k} \quad \text{if} \quad g_k^T d_k + d_k^T H_k d_k > 0$$
5: set
$$\widehat{\alpha}_k \leftarrow \frac{2(1-\eta)\Delta q(x_k, \tau_k, d_k)}{(\tau_k L_k + \Gamma_k)\|d_k\|_2^2} \quad \text{and}$$
$$\widetilde{\alpha}_k \leftarrow \widehat{\alpha}_k - \frac{4\|c_k\|_1}{(\tau_k L_k + \Gamma_k)\|d_k\|_2^2}$$
6: set
$$\alpha_k \leftarrow \begin{cases} \widehat{\alpha}_k & \text{if } \widehat{\alpha}_k < 1 \\ 1 & \text{if } \widetilde{\alpha}_k \leq 1 \leq \widehat{\alpha}_k \\ \widetilde{\alpha}_k & \text{if } \widetilde{\alpha}_k > 1 \end{cases}$$
7: set $x_{k+1} \leftarrow x_k + \alpha_k d_k$ and continue or update L_k and/or Γ_k and return to step 5
8: **end for**

Convergence theory

Exactly the same as for SQP-B, except different step size lower bound

- For SQP-A:

$$\alpha_k = \frac{2(1-\eta)\Delta q(x_k, \tau_k, d_k)}{(\tau_k L_k + \Gamma_k)\|d_k\|_2^2} \geq \frac{2(1-\eta)\kappa_q \tau_{\min}}{(\tau_{-1}\rho L + \rho\Gamma)\kappa_\Psi} > 0$$

- For SQP-B:

$$\alpha_k > \frac{2\nu(1-\eta)\Delta q(x_k, \tau_k, d_k)}{(\tau_k L + \Gamma)\|d_k\|_2^2} \geq \frac{2\nu(1-\eta)\kappa_q \tau_{\min}}{(\tau_{-1} L + \Gamma)\kappa_\Psi} > 0$$

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Stochastic setting

Consider the stochastic problem:

$$\min_{x\in\mathbb{R}^n} f(x) \equiv \mathbb{E}[F(x,\omega)] \quad \text{s.t. } c(x) = 0$$

Let us assume only the following:

Assumption

For all $k \in \mathbb{N}$, one can compute $\bar{g}_k$ with

$$\mathbb{E}_k[\bar{g}_k] = g_k \quad \textit{and} \quad \mathbb{E}_k[\|\bar{g}_k - g_k\|_2^2] \leq M$$

Search directions computed by:

$$\begin{bmatrix} H_k & J_k^T \\ J_k & 0 \end{bmatrix} \begin{bmatrix} \bar{d}_k \\ \bar{y}_k \end{bmatrix} = - \begin{bmatrix} \bar{g}_k \\ c_k \end{bmatrix}$$

Important: Given x_k, the values (c_k, J_k, H_k) are determined

Stochastic SQP with adaptive step sizes

(For simplicity, assume Lipschitz constants L and Γ are known.)

Algorithm : Stochastic SQP

choose $x_0 \in \mathbb{R}^n$, $\bar{\tau}_{-1} \in \mathbb{R}_{>0}$, $\sigma \in (0,1)$, $\{\beta_k\} \in (0,1]$

for $k \in \{0,1,2,\dots\}$ **do**

solve

$$\begin{bmatrix} H_k & J_k^T \\ J_k & 0 \end{bmatrix} \begin{bmatrix} \bar{d}_k \\ \bar{y}_k \end{bmatrix} = - \begin{bmatrix} \bar{g}_k \\ c_k \end{bmatrix}$$

set $\bar{\tau}_k$ to ensure $\Delta \bar{q}(x_k, \bar{\tau}_k, \bar{d}_k) \gg 0$, offered by

$$\bar{\tau}_k \leq \frac{(1-\sigma)\|c_k\|_1}{\bar{g}_k^T \bar{d}_k + \bar{d}_k^T H_k \bar{d}_k} \quad \text{if} \quad \bar{g}_k^T \bar{d}_k + \bar{d}_k^T H_k \bar{d}_k > 0$$

set

$$\bar{\hat{\alpha}}_k \leftarrow \frac{\beta_k \Delta \bar{q}(x_k, \bar{\tau}_k, \bar{d}_k)}{(\bar{\tau}_k L + \Gamma)\|\bar{d}_k\|_2^2} \quad \text{and}$$

$$\bar{\tilde{\alpha}}_k \leftarrow \bar{\hat{\alpha}}_k - \frac{4\|c_k\|_1}{(\bar{\tau}_k L + \Gamma)\|\bar{d}_k\|_2^2}$$

set

$$\bar{\alpha}_k \leftarrow \begin{cases} \bar{\hat{\alpha}}_k & \text{if } \bar{\hat{\alpha}}_k < 1 \\ 1 & \text{if } \bar{\tilde{\alpha}}_k \leq 1 \leq \bar{\hat{\alpha}}_k \\ \bar{\tilde{\alpha}}_k & \text{if } \bar{\tilde{\alpha}}_k > 1 \end{cases}$$

set $x_{k+1} \leftarrow x_k + \bar{\alpha}_k \bar{d}_k$

end for

step size control

The sequence $\{\beta_k\}$ allows us to consider, like for SG,

- a fixed step size
- diminishing step sizes (e.g., $\Theta(1/k)$)

Unfortunately, additional control on the step size is needed

- too small: insufficient progress
- too large: ruins progress toward feasibility / optimality

We never know when the step size is too small or too large!

Idea: Project $\bar{\widehat{\alpha}}_k$ and $\bar{\widetilde{\alpha}}_k$ onto

$$\left[\frac{\beta_k\bar{\tau}_k}{\bar{\tau}_k L + \Gamma}, \frac{\beta_k\bar{\tau}_k}{\bar{\tau}_k L + \Gamma} + \theta\beta_k^2\right]$$

where $\theta \in \mathbb{R}_{>0}$ is a user-defined parameter

Fundamental lemmas

Lemma

For all $k \in \mathbb{N}$, for any realization of $\bar{g}_k$, one finds

$$\begin{aligned} & \phi(x_k + \bar{\alpha}_k \bar{d}_k, \bar{\tau}_k) - \phi(x_k, \bar{\tau}_k) \\ \leq\ & \underbrace{-\bar{\alpha}_k \Delta q(x_k, \bar{\tau}_k, d_k)}_{\mathcal{O}(\beta_k),\text{ "deterministic"}} + \underbrace{\tfrac{1}{2}\bar{\alpha}_k \beta_k \Delta \bar{q}(x_k, \bar{\tau}_k, \bar{d}_k)}_{\mathcal{O}(\beta_k^2),\text{stochastic/noise}} + \underbrace{\bar{\alpha}_k \bar{\tau}_k g_k^T (\bar{d}_k - d_k)}_{\text{due to adaptive } \bar{\alpha}_k} \end{aligned}$$

Lemma

For all $k \in \mathbb{N}$, one finds

$$\mathbb{E}_k[\bar{d}_k] = d_k, \quad \mathbb{E}_k[\bar{y}_k] = y_k, \quad \text{and} \quad \mathbb{E}_k[\|\bar{d}_k - d_k\|_2] = \mathcal{O}(\sqrt{M})$$

as well as

$$g_k^T d_k \geq \mathbb{E}_k[\bar{g}_k^T \bar{d}_k] \geq g_k^T d_k - \zeta^{-1} M \quad \text{and} \quad d_k^T H_k d_k \leq \mathbb{E}_k[\bar{d}_k^T H_k \bar{d}_k]$$

Good merit parameter behavior

Lemma

If $\{\bar{\tau}_k\}$ eventually remains fixed at sufficiently small $\tau_{\min} > 0$, then for large k

$$\mathbb{E}_k[\bar{\alpha}_k \bar{\tau}_k g_k^T(\bar{d}_k - d_k)] = \beta_k^2 \tau_{\min} \mathcal{O}(\sqrt{M})$$

Theorem

If $\{\bar{\tau}_k\}$ eventually remains fixed at sufficiently small $\tau_{\min} > 0$, then for large k

$$\beta_k = \Theta(1) \implies \alpha_k = \frac{\tau_{\min}}{\tau_{\min} L + \Gamma} \implies \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{k} \Delta q(x_j, \tau_{\min}, d_j)\right] \leq \mathcal{O}(M)$$

$$\beta_k = \Theta\left(\frac{1}{k}\right) \implies \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^{k}\beta_j\right)}\sum_{j=1}^{k} \beta_j \Delta q(x_j, \tau_{\min}, d_j)\right] \to 0$$

Good merit parameter behavior

Lemma

If $\{\bar{\tau}_k\}$ eventually remains fixed at sufficiently small $\tau_{\min} > 0$, then for large k

$$\mathbb{E}_k[\bar{\alpha}_k \bar{\tau}_k g_k^T(\bar{d}_k - d_k)] = \beta_k^2 \tau_{\min} \mathcal{O}(\sqrt{M})$$

Theorem

If $\{\bar{\tau}_k\}$ eventually remains fixed at sufficiently small $\tau_{\min} > 0$, then for large k

$$\beta_k = \Theta(1) \implies \alpha_k = \frac{\tau_{\min}}{\tau_{\min} L + \Gamma} \implies \mathbb{E}\left[\frac{1}{k}\sum_{j=1}^{k}(\|g_j + J_j^T y_j\|_2 + \|c_j\|_2)\right] \leq \mathcal{O}(M)$$

$$\beta_k = \Theta\left(\frac{1}{k}\right) \implies \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^{k}\beta_j\right)}\sum_{j=1}^{k}\beta_j(\|g_j + J_j^T y_j\|_2 + \|c_j\|_2)\right] \to 0$$

Poor merit parameter behavior

$\{\bar{\tau}_k\} \searrow 0$:

- cannot occur if $\|\bar{g}_k - g_k\|_2$ is bounded uniformly
- occurs with small probability if distribution of $\bar{g}_k$ has *fast* decay

$\{\bar{\tau}_k\}$ remains too large:

- can only occur if realization of $\{\bar{g}_k\}$ is *one-sided for all k*
- if there exists $p \in (0, 1]$ such that, for all k in infinite $\mathcal{K}$,

$$\mathbb{P}_k\left[\bar{g}_k^T \bar{d}_k + \max\{\bar{d}_k^T H_k \bar{d}_k, 0\} \geq g_k^T d_k + \max\{d_k^T H_k d_k, 0\}\right] \geq p$$

then occurs with probability zero

Numerical results

Matlab software: https://github.com/frankecurtis/StochasticSQP

CUTE problems with noise added to gradients with different noise levels

- Stochastic SQP: 10^3 iterations
- Stochastic Subgradient: 10^4 iterations and tuned over 11 values of τ

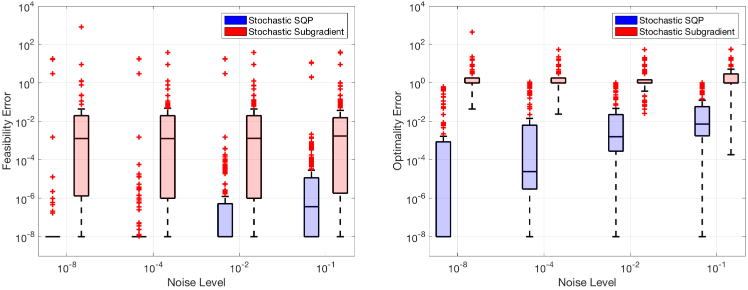

Figure: Box plots for feasibility errors (left) and optimality errors (right).

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Complexity of deterministic algorithm

All reductions in the merit function can be cast in terms of smallest τ.

Lemma 7

If $\{\tau_k\}$ eventually remains fixed at sufficiently small $\tau_{\min}$, then for any $\epsilon \in (0,1)$ there exists $(\kappa_1, \kappa_2) \in (0,\infty) \times (0,\infty)$ such that, for all k,

$$\|g_k + J_k^T y_k\| > \epsilon \text{ or } \sqrt{\|c_k\|_1} > \epsilon \implies \Delta q(x_k, \tau_k, d_k) \geq \min\{\kappa_1, \kappa_2 \tau_{\min}\}\epsilon.$$

Since $\tau_{\min}$ is determined by the initial point, *it will be reached.*

Theorem 8

For any $\epsilon \in (0,1)$, there exists $(\kappa_1, \kappa_2) \in (0,\infty) \times (0,\infty)$ such that

$$\|g_k + J_k^T y_k\| \leq \epsilon \text{ and } \sqrt{\|c_k\|_1} \leq \epsilon$$

in a number of iterations no more than

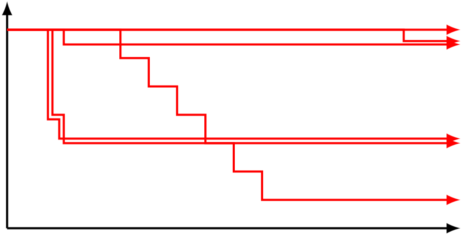
$$\left(\frac{\tau_{-1}(f_0 - f_{\inf}) + \|c_0\|_1}{\min\{\kappa_1, \kappa_2 \tau_{\min}\}}\right)\epsilon^{-2}.$$

Challenge in the stochastic setting

We are minimizing a function that is changing during the optimization.

In the stochastic setting, minimum τ is not determined by the initial point.

- Even if we assume $\tau_k \geq \tau_{\min} > 0$ for all k in any realization, the final value of the merit parameter τ is not determined.
- This means we cannot cast all reductions in terms of some fixed τ.



Our approach

In fact, τ reaching some minimum value is not necessary.

- Important: Diminishing probability of continued imbalance between "true" merit parameter update and "stochastic" merit parameter update.
- In iteration k, the algorithm has obtained the merit parameter value $\bar{\tau}_{k-1}$.
- If the true gradient is computed, then one obtains $\tau_k^{\text{trial,true}}$.
- Consider the random index set

$$\mathcal{K}_\tau := \{k : \tau_k^{\text{trial,true}} < \bar{\tau}_{k-1}\}.$$

Lemma 9

For any $\delta \in (0,1)$, one finds that

$$\mathbb{P}\left[|\mathcal{K}_\tau| \leq \left\lceil \frac{\ell(s_{\max},\delta)}{p} \right\rceil\right] \geq 1-\delta,$$

where

$$\ell(s,\delta) := s + \log(1/\delta) + \sqrt{\log(1/\delta)^2 + 2s\log(1/\delta)} > 0.$$

Chernoff bound

How do we get there?

Lemma 10 (Chernoff bound, multiplicative form)

For any k, let $\{Y_0, \dots, Y_k\}$ be independent Bernoulli random variables. Then, for any $s_{\max} \in \mathbb{N}$ and $\delta \in (0,1)$,

$$\sum_{j=0}^{k} \mathbb{P}[Y_j = 1] \geq \ell(s_{\max}, \delta) \implies \mathbb{P}\left[\sum_{j=0}^{k} Y_j \leq s_{\max}\right] \leq \delta.$$

We construct a tree whose nodes are signatures of possible runs of the algorithm.

- A realization $\{\bar{g}_0, \dots, \bar{g}_k\}$ belongs to a node if and only if a certain number of decreases of τ have occurred and the probability of decrease in the current iteration is in a given closed/open interval.
- Bad leaves are those when the probability of decrease has accumulated beyond a threshold, yet the merit parameter has not been decrease sufficiently often.
- Along the way, we apply a Chernoff bound on a carefully constructed set of random variables to bound probabilities associated with bad leaves.

Worst-case iteration complexity of $\widetilde{\mathcal{O}}(\epsilon^{-4})$

Theorem 11

Suppose the algorithm is run

- $k_{\max}$ *iterations with*
- $\beta_k = \gamma/\sqrt{k_{\max}+1}$ *and*
- *the merit parameter is reduced at most* $s_{\max} \in \{0, 1, \dots, k_{\max}\}$ *times.*

Let k_* *be sampled uniformly over* $\{1, \dots, k_{\max}\}$*. Then, with probability* $1-\delta$*,*

$$\mathbb{E}[\|g_{k_*} + J_{k_*}^T y_{k_*}\|_2^2 + \|c_{k_*}\|_1] \leq \frac{\tau_{-1}(f_0 - f_{\inf}) + \|c_0\|_1 + M}{\sqrt{k_{\max}+1}} + \frac{(\tau_{-1} - \tau_{\min})(s_{\max}\log(k_{\max}) + \log(1/\delta))}{\sqrt{k_{\max}+1}}$$

Theorem 12

If the stochastic gradient estimates are sub-Gaussian, then w.p. $1-\bar{\delta}$

$$s_{\max} = \mathcal{O}\left(\log\left(\log\left(\frac{k_{\max}}{\bar{\delta}}\right)\right)\right).$$

Outline

Motivation

SG and SQP

Adaptive (Deterministic) SQP

Stochastic SQP

Worst-Case Iteration Complexity

Extensions

Conclusion

Recent work (under review): No LICQ

Remove constraint qualification

- infeasible and/or degenerate problems
- step decomposition method

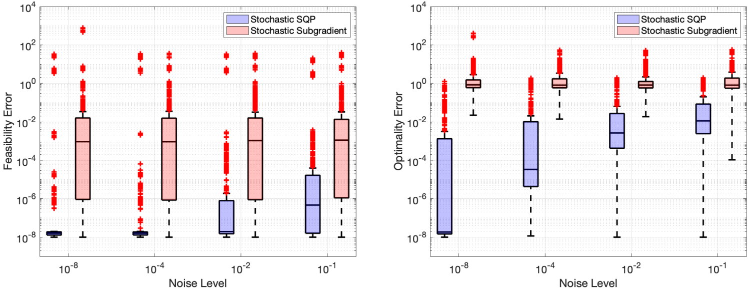


Figure: Box plots for feasibility errors (left) and optimality errors (right).

Recent work (under review): Matrix-free methods

Inexact subproblem solves

- stochasticity and inexactness(!)
- applicable for large-scale, e.g., PDE-constrained

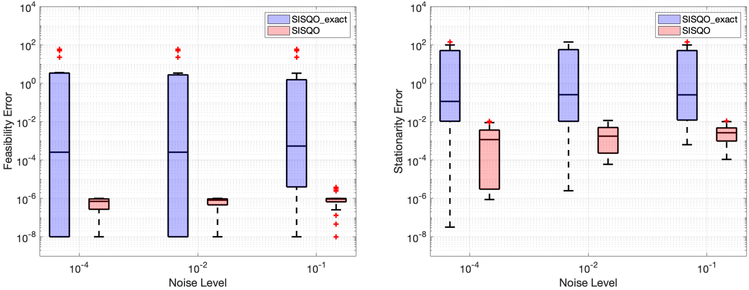


Figure: Box plots for feasibility errors (left) and optimality errors (right).

Current work: Inequality constraints

Inequality constraints

- SQP
- interior-point

Main challenge: For *equality* constraints only, subproblem solution on linearized constraints remains unbiased:

$$c_k + J_k \bar{d}_k = 0 \iff \bar{d}_k = v_k + \bar{u}_k$$

$$\text{with} \quad v_k \in \text{Range}(J_k^T) \quad \text{and} \quad \bar{u}_k \in \text{Null}(J_k)$$

$$\text{has} \quad E_k[\bar{u}_k] = u_k.$$

However, when *inequalities* are present, subproblem solution is biased.

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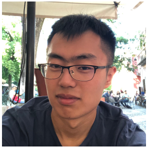
Summary

Consider *equality constrained* stochastic optimization:

$$\min_{x \in \mathbb{R}^n} f(x) \equiv \mathbb{E}[F(x,\omega)]$$
$$\text{s.t. } c_{\mathcal{E}}(x) = 0$$

- *Adaptive* SQP method for deterministic setting
- *Stochastic* SQP method for stochastic setting
- Convergence in expection (comparable to SG for unconstrained setting)
- Worst-case complexity on par with stochastic subgradient method
- Numerical experiments are very promising
- Various extensions (on-going)

Collaborators and references

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