A corrected proof of Lemma 4.9 from the published paper is provided below.

LEMMA 4.9. *Suppose Assumption 4.7 holds. Then, the sequences* $\{\|\overline{d}_k\|\}$ *and* $\{\|\widehat{d}_k\|\}$ *are bounded above, so the sequence* $\{\|d_k\|\}$ *is bounded above.*

Proof. Under Assumption 4.7, there exists $\tau > 0$ such that $v(x_k) \leq \tau$ and $\|\nabla f(x_k)\| \leq \tau$ for any k. In order to derive a contradiction to the statement in the lemma, suppose that $\{\|\overline{d}_k\|\}$ is not bounded. Then, there exists an iteration k yielding $\|\overline{d}_k\|^2 > 2\tau/\underline{\mu} =: \overline{\tau}$. The objective value of subproblem (3.7) corresponding to this $\overline{d}_k$ satisfies

$$l(\overline{d}_k; x_k) + \tfrac{1}{2}\overline{d}_k^T H(x_k, 0, \overline{\lambda}_k)\overline{d}_k \geq \tfrac{1}{2}\underline{\mu}\|\overline{d}_k\|^2 > \tau \geq v(x_k).$$

However, this is a contradiction as $v(x_k)$ is the objective value corresponding to $(d, r, s, t) = (0, [c^{\mathcal{E}}(x_k)]^+, [c^{\mathcal{E}}(x_k)]^-, [c^{\mathcal{I}}(x_k)]^+)$, which is also feasible for this subproblem. Thus, $\|\overline{d}_k\|^2 \leq \overline{\tau}$ for all k, so $\{\|\overline{d}_k\|\}$ is bounded. Observe that by the optimality of $(\overline{d}_k, \overline{r}_k, \overline{s}_k, \overline{t}_k)$ for (3.7), we also have

$$e^T(\overline{r}_k + \overline{s}_k) + e^T\overline{t}_k \leq e^T(\overline{r}_k + \overline{s}_k) + e^T\overline{t}_k + \tfrac{1}{2}\overline{d}_k^T H(x_k, 0, \overline{\lambda}_k)\overline{d}_k \leq v(x_k). \qquad (\star)$$

Now suppose, in order to derive a different contradiction, that for some k the optimal solution $(\widehat{d}_k, \widehat{r}_k, \widehat{s}_k, \widehat{t}_k)$ for (3.9) has

$$\|\widehat{d}_k\| > \frac{\rho_0\tau + \sqrt{(\rho_0\tau)^2 + 2\underline{\mu}(\rho_0\tau\overline{\tau} + \tau + \frac{1}{2}\overline{\mu}\overline{\tau}^2)}}{\underline{\mu}}. \qquad (\star\star)$$

Under Assumption 4.7, we have from $(\star)$ and $(\star\star)$ that

$$\begin{aligned}
&\widehat{\rho}_k \nabla f(x_k)^T \widehat{d}_k + e^T(\widehat{r}_k^{\mathcal{E}_k^c} + \widehat{s}_k^{\mathcal{E}_k^c}) + e^T\widehat{t}_k^{\mathcal{I}_k^c} + \tfrac{1}{2}\widehat{d}_k^T H(x_k, \widehat{\rho}_k, \widehat{\lambda}_k)\widehat{d}_k \\
&\quad - \overline{\rho}_k \nabla f(x_k)^T \overline{d}_k - e^T(\overline{r}_k^{\mathcal{E}_k^c} - \overline{s}_k^{\mathcal{E}_k^c}) - e^T\overline{t}_k^{\mathcal{I}_k^c} - \tfrac{1}{2}\overline{d}_k^T H(x_k, \widehat{\rho}_k, \widehat{\lambda}_k)\overline{d}_k \\
\geq &\ -\rho_0\|\nabla f(x_k)\|(\|\widehat{d}_k\| + \|\overline{d}_k\|) - e^T(\overline{r}_k + \overline{s}_k) - e^T\overline{t}_k + \tfrac{1}{2}\underline{\mu}\|\widehat{d}_k\|^2 - \tfrac{1}{2}\overline{\mu}\|\overline{d}_k\|^2 \\
\geq &\ -\rho_0\tau(\|\widehat{d}_k\| + \overline{\tau}) - \tau + \tfrac{1}{2}\underline{\mu}\|\widehat{d}_k\|^2 - \tfrac{1}{2}\overline{\mu}\overline{\tau}^2 > 0,
\end{aligned}$$

contradicting the optimality of $(\widehat{d}_k, \widehat{r}_k, \widehat{s}_k, \widehat{t}_k)$ for (3.9). Thus, $\{\|\widehat{d}_k\|\}$ is also bounded.

The boundedness of $\{\|d_k\|\}$ follows from the above results and the fact that d_k is chosen as a convex combination of $\overline{d}_k$ and $\widehat{d}_k$ for all k. ☐