Decomposition Part, Week 1

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COR@L
COMPUTATIONAL OPTIMIZATION RESEARCH AT LEHIGH

01/26/07

We will Cover:

- Dantzig-Wolfe decomposition and column generation in IP.
- 2 applications of DW decomposition: maximum independent set problem, multistage stochastic IP.
- Lagrangian relaxation (this week or next week).
- Formulation of Benders decomposition.
- Global optimization of nonconvex MINP with decomposable structures.
- Any suggestions? Additions? Not interesting?
- Volunteers?
- Suggestions for the structure of the seminar?

Decomposition In General

1. DW Decomposition
2. Lagrangian relaxation
3. Benders decomposition

- All are large scale (models with large number of constraints/variables) optimization algorithms.
- We can find better bounds for branch and bound using these approaches.
- For the first 2, alternative approaches are cutting plane, variable redefinition.
- Basically, first 2 decomposition includes 3 steps:
  - Decompose system of inequalities into two parts.
  - Find the convex hull of the second system.
  - optimize first system over this convex hull.

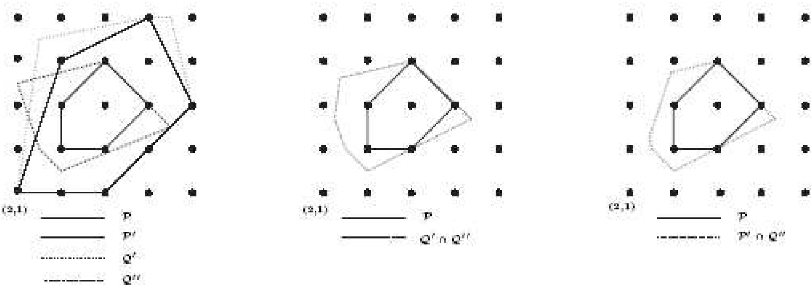
Why decomposition can find a better bound?

- From T. Ralphs and M. Galati, Decomposition in Integer Programming, in Integer Programming: Theory and Practice, John Karlof, ed. (2005), 57.

$$\begin{aligned} &\min cx \\ \text{s.t.} &Ax \le b \\ &x \in \mathbb{Z}^n \end{aligned}$$

- Let $Q = \{x \in \mathbb{R}^n | Ax \le b\} = \{x \in \mathbb{R}^n | A^{''}x \le b^{''}, A^{'}x \le b^{'}\}$.
- Let $F = Q \cap \mathbb{Z}^n$ (feasible integer points) and P be the convex hull of F.
- Let $Q^{'} = \{x \in \mathbb{R}^n | A^{'}x \le b^{'}\}$, $F^{'} = Q^{'} \cap \mathbb{Z}^n$ (feasible integer points) and $P^{'}$ be the convex hull of $F^{'}$.
- Let $Q^{''} = \{x \in \mathbb{R}^n | A^{''}x \le b^{''}\}$.
- $z_{LP} = min_{x \in Q}\{cx\}$.
- $z_{DECOMP} = min_{x \in P^{'}}\{cx | A^{''}x \le b^{''}\} = min_{x \in P^{'} \cap Q^{''}}\{cx\}$.

Why decomposition can find a better bound?



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Source: T. Ralphs and M. Galati, Decomposition in Integer Programming, in Integer Programming: Theory and Practice, John Karlof, ed. (2005), 57.

Dantzig Wolfe Decomposition

- Assume $P^{'}$ is bounded. Let $E \subseteq F^{'}$ be the set of *extreme* points of $P^{'}$.

$$\mathcal{P}' = \{x \in \mathbb{R}^n \mid x = \sum_{s \in \mathcal{E}} s\lambda_s, \sum_{s \in \mathcal{E}} \lambda_s = 1, \lambda_s \geq 0 \; \forall s \in \mathcal{E}\}.$$

- Dantzig Wolfe formulation and bound:

$$z_{DW} = \min_{x \in \mathbb{R}^n} \{c^\top x \mid A''x \geq b'', x = \sum_{s \in \mathcal{E}} s\lambda_s, \sum_{s \in \mathcal{E}} \lambda_s = 1, \lambda_s \geq 0 \; \forall s \in \mathcal{E}\}.$$

- Substitute new variables:

$$z_{DW} = \min_{\lambda \in \mathbb{R}_+^{\mathcal{E}}} \{c^\top (\sum_{s \in \mathcal{E}} s\lambda_s) \mid A''(\sum_{s \in \mathcal{E}} s\lambda_s) \geq b'', \sum_{s \in \mathcal{E}} \lambda_s = 1\}.$$

- $|E|$ may be very large, thus E should be generated dynamically.

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Source: T. Ralphs and M. Galati, Decomposition in Integer Programming, in Integer Programming: Theory and Practice, John Karlof, ed. (2005), 57.

Cutting Strips Problem

$$[P^{cs}] \quad \begin{aligned} \min \quad & \sum_{k=1}^{K} L^k \left(W^k y^k - \sum_{i=1}^{p} w_i z_i^k \right) \\ \text{s.t.} \quad & \\ & \sum_{k=1}^{K} L^k z_i^k \geqslant d_i, \quad i = 1, \ldots, p, \end{aligned} \qquad (2)$$

$$\sum_{i=1}^{p} w_i z_i^k \leqslant W^k y^k, \quad k = 1, \ldots, K, \qquad (3)$$

$$y^k \in \{0, 1\}, \quad k = 1, \ldots, K, \qquad (4)$$

$$z_i^k \in \mathbb{N}, \quad i = 1, \ldots, p, \; k = 1, \ldots, K. \qquad (5)$$

Source:Vanderbeck F.,On Dantzig-Wolfe Decomposition in Integer Programming and ways to Perform Branching in a Branch-and-Price Algorithm, Oper. Res. 48-1, 2000, pg. 111.

Integer Points in Second System

- Let Q^k be the set of feasible cutting patterns for sheet k, λ_q^k be the number of times pattern q^k is selected.

$$X^{cs} = \left\{ (y^k, z^k)_{k=1,\dots,K} \in \mathbb{N}^{K(1+p)} : \right.$$

$$\left. \sum_{i=1}^{p} w_i z_i^k \leqslant W^k y^k \text{ for } k = 1, \dots, K \right\}$$

$$= \left\{ (y^k, z^k)_{k=1,\dots,K} \in \mathbb{R}^{K(1+p)} : \right.$$

$$y^k = \sum_{q^k \in Q^k} \lambda_q^k,\ z^k = \sum_{q^k \in Q^k} q^k \lambda_q^k,$$

$$\left. \sum_{q^k \in Q^k} \lambda_q^k \leqslant 1 \ \forall k,\ \ \lambda_q^k \in \{0,1\} \ \forall q^k \in Q^k, k \right\}.$$

Dantzig Wolfe Decomposition of Cutting Strips Problem

$$\min \quad \sum_{k=1}^{K} \sum_{q^k \in Q^k} c_q^k \lambda_q^k$$

$[M^{cs}]$ s.t.

$$\sum_{k=1}^{K} \sum_{q^k \in Q^k} L^k q_i^k \lambda_q^k \geqslant d_i, \quad i = 1, \ldots, p, \qquad (6)$$

$$\sum_{q^k \in Q^k} \lambda_q^k \leqslant 1, \quad k = 1, \ldots, K,$$

$$\lambda_q^k \in \{0, 1\}, \quad q^k \in Q^k, k = 1, \ldots, K.$$

Column Generation

- Master problem may include infinitely many cutting patterns:

$$Q^k = \{q^k \in \mathbb{N}^p : \sum_i^p w_i q_i^k \leq W\}$$

- Start with “a” set of feasible cutting patterns → RMP.
- Generate the column with most negative reduced cost to improve the bound.
- Let π_i be the dual variable for demand constraint i, μ_k be the dual variable for convex comb. constr. Subproblem:

$$\zeta^k(\pi,\mu) = \min \quad L^k \left(W^k - \sum_{i=1}^{p} (w_i + \pi_i) z_i \right) + \mu_k$$

s.t.

$$\sum_{i=1}^{p} w_i z_i \leq W^k,$$

$$z_i \in \mathbb{N}, \quad i = 1, \ldots, p,$$

Lower bound for DW Decomposition LP

If column generation subproblem is solved to optimality:

$$\begin{aligned} z_{DW}(LP)^{LB} &= \text{(obj. value of dual master)} \\ &+ \text{(obj. function value of optimal column generation subproblem)} \end{aligned}$$

Let q be the optimal solution to the column generation problem.

$$z_{DW}(LP)^{LB} = (\sum_{i=1..p} d_i \pi_i + \sum_{k=1..K} \mu_k) + (\sum_{k=1..K} c_q q_k - \sum_{k=1..K, i=1..p} L^k q_i^k - \sum_{k=1..K} \mu_k)$$

Some Points

- Lagrangian relaxation and Dantzig Wolfe decomposition is used to find better bounds for the MIP.
- Lagrangian relaxation, Dantzig Wolfe decomposition, cutting plane, variable redefinition can be used to get the same bound.
- Cutting plane: convex hull of second system $P^{'}$ is defined by finding the facet defining inequalities of the system.
- Master LP (from Dantzig Wolfe Decomp.) = dual formulation of the Lagrangian dual that results from dualizing the $A^{''}x \leq b^{''}$.
- Dantzig Wolfe decomposition leads to models with large number of variables which requires column generation algorithm.
- Variable redefinition: develop an alternative formulation Z for the polyhedron $P^{'}$.

Discretization vs. Convexification

- The first DW example is convexification.
- Cutting Strip problem uses discretization.
- Convexification: $\lambda_s > 0$,
- Discretization: $\lambda_s \in \{0, 1\}$ if it is an extreme point, $\lambda_s \in \mathbb{N}$ if it is an extreme ray.
- Convexification: $x = \sum_s s\lambda_s$ does not imply x is integer. To express integrality, must return to original variables. Branching should be in original variables.
- Discretization: $x = \sum_s s\lambda_s$ results integer variables. Can do branching or write cuts in terms of λ_s.
- Both give the same LP relaxation of master.
- Both has the same IP master if variables are binary.
- Discretization $\neq$ convexification if some variables are general integer.

Section 1 (Vanderbeck 2000): Decomposition of IP

- General form of master problem.
- $\sum_s \lambda_s = 1$ can be changed with $\sum_s \lambda_s \leq 1$ if 0 vector is a feasible solution for the problem.
- Independent subsystems: $Dx \leq d \rightarrow D^k x \leq d^k, k = 1, .., K.$

$$\sum_{q \in Q(k)} \lambda_q = 1, k = 1, .., K.$$

- Identical independent subsystems: $Dx \leq d \rightarrow D^k x \leq d^k, k = 1, .., K$, but $D^k = \bar{D}$, $d^k = \bar{d}$ for $k = 1..K$.

$$\sum_{q \in \bar{Q}} \lambda_q = K.$$

Equivalence Between Original and Master Formulation

- Let P be the original problem, M be the master problem.
- Both have the same set of feasible integer points.
- Since the representation of solution is different in both formulations, a solution x for P may not result a unique solution λ for M.
- Cases when the solutions do not corresponds to a unique solution is given in the paper.
- $Z_{LP}(P) \leq Z_{LP}(M) \leq Z_{IP}$.
- $Z_{LP}(P) < Z_{LP}(M)$ if subsystem does not have integrality property.

Practical Issues with Column Generation

- At each branch and bound node, a feasible LP solution is required. Especially with branching it becomes difficult to maintain feasibility. One way is to use artificial variables that will always result feasible solutions.
- Designing branching rules: Branching rule must be incorporated into subproblem (column generation problem).
- **RMP:**Optimize RMP to get dual variables. Dual solution is not unique if primal is degenerate. Dual variables effect the generated column.
  - Initialization is necessary with simplex and for other methods it can reduce heading-in effect of initially producing bad duals which may cause irrelevant columns.
  - Solution with primal, primal-dual simplex, barrier?
  - Simplex based column generation has tailing off effect (poor convergence).