Valid Inequalities for MILPs

Cor@l Seminar Series

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Review

Recall, for the polyhedron

$$P = \left\{x \in \mathbb{R}_+^{n+p} \mid Ax \geq b\right\}$$

and the mixed $0, 1$ set

$$S = \left\{x \in \mathbb{B}^n \times \mathbb{R}_+^p \mid Ax \geq b\right\},$$

the *lift-and-cut* procedure (Balas et al):

1: Select $j \in \{1, \ldots, n\}$.
2: Generate the nonlinear system $x_j(Ax - b) \geq 0$, $(1 - x_j)(Ax - b) \geq 0$.
3: Linearize the system by substituting y_i for $x_i x_j, i \neq j$, and x_j for x_j^2. Call this polyhedron M_j.
4: Project M_j onto the x-space. Let P_j be the resulting polyhedron.

Optimizing over P_j

Goal: To find a valid inequality for the relaxation P_j that cuts off a given point $\overline{x}$

- We can write M_j as

$$M_j := \Big\{x \in \mathbb{R}^{n+p}_+, y \in \mathbb{R}^{n+p-1}_+ \mid A_j y + (a^j - b)x_j \geq 0, Ax + (b - a^j)x_j - A_j y$$

- Renaming the coefficient matrices of x yields the simpler form

$$M_j := \Big\{x \in \mathbb{R}^{n+p}_+, y \in \mathbb{R}^{n+p-1}_+ \mid \overline{B}_j x + A_j y \geq 0, \overline{A}_j x - A_j y \geq b\Big\}.$$

- To project on x-space (using Theorem 2), the appropriate cone is

$$Q := \{(u, v) \mid uA_j - vA_j = 0, u \geq 0, v \geq 0\}.$$

The Lift-and-project Cut

This yields

$$P_j = \left\{ x \in \mathbb{R}^{n+p}_{+} \mid (u\overline{B}_j + v\overline{A}_j)x \geq vb \text{ for all } (u,v) \in Q \right\}.$$

Now, given fractional solution $\overline{x}$, the inequality $\alpha x \geq \beta$ is valid for P_j, where $\alpha = u\overline{B}_j + v\overline{A}_j$ and $\beta = vb$, and $\alpha\overline{x} < \beta$ cuts off $\overline{x}$.

To get the deepest cut, we can solve the *cut generating LP*

$$\begin{aligned} \max \quad & vb - (u\overline{B}_j + v\overline{A}_j)\overline{x} \\ \text{subject to} \quad & uA_j - vA_j \ = \ 0 \\ & u, v \ \geq \ 0. \end{aligned}$$

But, this is a very large LP to solve for a single cut...

An Alternative Derivation

By Theorem 5, P_j is the convex hull of the union of two polyhedra:

$$\begin{array}{rcl} Ax & \geq & b \\ x & \geq & 0 \\ -x_j & \geq & 0 \end{array} \quad \text{and} \quad \begin{array}{rcl} Ax & \geq & b \\ x & \geq & 0 \\ x_j & \geq & 1 \end{array}.$$

Then, by Theorem 4,

$$P_j = \text{proj}_x \left\{ \begin{array}{lcr} Ax^0 & \geq & by_0 \\ -x_j^0 & \geq & 0 \\ Ax^1 & \geq & by_1 \\ x_j^1 & \geq & y_1 \\ x^0 + x^1 & = & x \\ y_0 + y_1 & = & 1 \\ x, x^0, x^1, y_0, y_1 & \geq & 0 \end{array} \right. .$$

Another Cut Generating LP

Ignoring some details, this leads to the cut generating LP

$$
\begin{aligned}
\text{min} \quad & \alpha\bar{x} - \beta \\
\text{subject to} \quad & \alpha - uA + u_0 e_j \ \geq \ 0 \\
& \alpha - vA - v_0 e_j \ \geq \ 0 \\
& \beta - ub \ \leq \ 0 \\
& \beta - vb - v_0 \ \leq \ 0 \\
& \sum_{i=1}^{m} u_i + u_0 + \sum_{i=1}^{m} v_i + v_0 \ = \ 1 \\
& u, u_0, v, v_0 \ \geq \ 0
\end{aligned}
\tag{1}
$$

There is a direct correspondence between basic feasible solutions of (1) and the basic solutions of the usual LP relaxation (Balas and Perregaard):

$$\min\{cx \mid Ax \geq b, x \geq 0\}. \tag{R}$$

Strengthening the Cuts

The integrality of the other variables can be used to strengthen the cuts found by solving (1).

Theorem 1. [Balas and Jeroslow] *Let $\overline{x}$ satisfy $Ax \geq b, x \geq 0$. Given an optimal solution u, u_0, v, v_0 of the cut generating LP* (1)*, define $m_k = \frac{va^k - ua^k}{u_0+v_0}$,*

$$a_k = \begin{cases} \min(ua^k + u_0\lceil m_k \rceil, va^k + v_0\lceil m_k \rceil) & \text{for } k = 1, \ldots, n \\ \max(ua^k, va^k) & \text{for } k = n+1, \ldots, n+p \end{cases},$$

and $\beta = \min(ub, vb + u_0)$. Then, the inequality $\alpha x \geq \beta$ is valid for $\mathrm{conv}(S)$.

Proof is same derivation as before, with a more general disjunction.

Lather, Rinse, Repeat

Theorem 2. [Balas] $P_n(P_{n-1}(\ldots P_2(P_1)\ldots)) = \mathrm{conv}(S)$

- The *rank* of p is the smallest integer such that $P^k = \mathrm{conv}(S)$.
- This shows that the lift-and-project rank of P is at most n.
- This is the best we can hope for, in general (Cook and Dash, Goemans and Tunçel).

Gomory's Derivation

Consider the mixed integer linear set defined by the equality constraint

$$S = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p \mid \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b\}.$$

Let

$$b = \lfloor b \rfloor + f_0 \quad \text{where } 0 < f_0 < 1$$
$$a_j = \lfloor a_j \rfloor + f_j \quad \text{where } 0 \le f_j < 1.$$

Then

$$\sum_{f_j \le f_0} f_j x_j + \sum_{f_j > f_0} (f_j - 1) x_j + \sum_{j=1}^{n} g_j y_j = k + f_0.$$

where k is some integer.

A Valid Disjunction

Since we must have $k \leq -1$ or $k \geq 0$, any $x \in S$ satisfies the disjunction

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{1 - f_j}{f_0} x_j + \sum_{j=1}^{n} \frac{g_j}{f_0} y_j \geq 1 \tag{2}$$

OR

$$-\sum_{f_j \leq f_0} \frac{f_j}{1 - f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j - \sum_{j=1}^{n} \frac{g_j}{1 - f_0} y_j \geq 1, \tag{3}$$

which follows from simple substitution and simplification. This is of the form $a^1 z \geq 1$ or $a^2 z \geq 1$, which implies $\sum_j \max(a_j^1, a_j^2) z_j \geq 1$ for *any* $z \geq 0$.

Gomory Mixed Integer Inequality

So, for each variable, we need the maximum coefficent in (2) and (3). We can find this easily, since one is positive and one is negative. This yields

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1 - f_0} y_j \ge 1. \qquad \text{(GMI)}$$

which must be valid for S. This is the *Gomory mixed integer inequality* (GMI inequality).

- This procedure can be generalized to the case with more than one constraint.

Take That Chvátal

- In the pure integer case, (GMI) reduces to

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j \ge 1.$$

Since $\frac{1-f_j}{1-f_0} < \frac{f_j}{f_0}$ when $f_j > f_0$, the GMI inequality dominates the *fractional cut*

$$\sum_{j=1}^{n} f_j x_j \ge f_0 \tag{4}$$

from Chvátal's procedure.

GMI Inequality Generation

- Unlike lift-and-project cuts, given a point $(\overline{x}, \overline{y}) \in P \setminus S$, it is $\mathcal{NP}$-hard to find a GMI cut or show that none exists.
- But, if $(\overline{x}, \overline{y}) \in P \setminus S$ is *basic*, this is not the case.
- Any row of the simplex tableau for which $\overline{x}_j$ is fractional for some j can be used to generate a cut of this from.

GMI Example

Consider the MILP

$$\begin{aligned} \max \quad & x + 2y \\ \text{subject to} \quad & -x + y \leq 2 \\ & x + y \leq 5 \\ & 2x - y \leq 4 \\ & x \in \mathbb{Z}_+, y \in \mathbb{R}_+ \end{aligned}$$

Adding slacks and solving the LP relaxation yields the optimal tableau

$$\begin{array}{ccccccccc} z & & & +0.5s_1 & +1.5s_2 & & = & 8.5 \\ & & y & +0.5s_1 & +0.5s_2 & & = & 3.5 \\ & x & & -0.5s_1 & +0.5s_2 & & = & 1.5 \\ & & & 0.5s_1 & -0.5s_2 & +s_3 & = & 4.5 \end{array}$$

and the (fractional) solution $(\overline{x}, \overline{y}) = (1.5, 3.5)$.

GMI Example (cont.)

Using the row in which x is basic,

$$x - 0.5s_1 + 0.5s_2 = 1.5,$$

where $f_0 = 0.5$, and applying the GMI procedure yields the cut

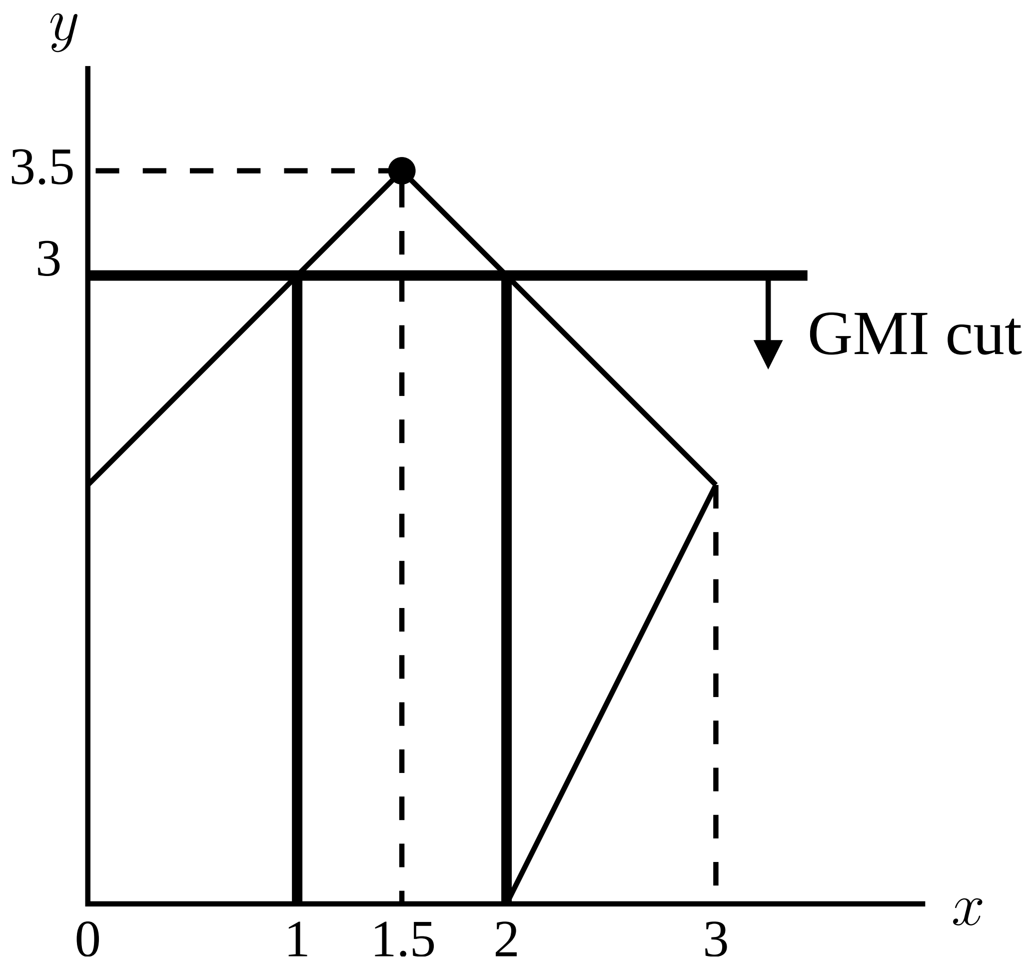
$$s_1 + s_2 \geq 1.$$

Using $s_1 + s_2 = 7 - 2y$ (from the initial tableau) gives the cut in the original space:

$$y \leq 3.$$

A Thousand Words

The GMI example is illustrated in the figure below.



Bringing It All Together

In the $0, 1$ world...

- The GMI cuts can be improved using the correspondence between the lift-and-project cut generating LP (1) and the usual LP relaxation (R).
  - There is an equivalence between GMI cuts generated from (R) and strengthened lift-and-project cuts generated from (1).
  - Using this equivalence we can improve a GMI cut by pivoting if the corresponding columns in lift-and-project tableau have negative reduced costs.
- Empirical evidence shows that the GMI cuts can be improved about %75 of the time.