Selected Topics in Column Generation

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February 1, 2007

Choosing a solver for the Master

- Solve in the dual space(Kelly's method) by applying a cutting plane algorithm
- In the bundle method(Lemarechal), a quadratic term that penalizes the norm of the deviation to the current best dual solution to stabilize the search
- Apply Interior point method to the dual of RMP, it benefits by computing dual prices centered in the optimal face when dual admits alternative optimum solutions(also helps stabilization)
- A sub-gradient algorithm, finds approximate dual solution, uses little computation, may violate the master constraints

- Simplex method leads to an optimal basis at random, thus It can produce complementary kinds of columns
- Interior point methods produce a solution in the relative interior of optimal face
- An approximation solution to RMP may suffice
- Volume algorithm(extension of subgradient algorithm) rapidly produces primal and dual approximate solutions
- Uses volumes below the active face to compute the dual variable values
- Advantages of the volume algorithm; small memory requirements, numerical stability, and fast convergence
- The solution method may be dynamically switched during the whole process
- Heuristics can be used to construct or improve dual variable at any time

Dual point of view, Kelly's method

- The dual of the RMP is the dual master problem.
- The pricing problem for RMP is a separation problem for the dual.
- Consider a pair of feasible and bounded primal and dual problems
  $min\left\{c^T\lambda | A\lambda = b, \lambda \geq 0\right\}$ and $max\left\{b^Tu | A^Tu \leq= c\right\}$
- Structural inequalities $F^Tu \leq f$ are added to the dual at initialization time, assuming these inequalities do not change optimal value
- Then, extended forms are

$$min\left\{c^T\lambda + f^Ty | A\lambda + Fy = b, \lambda, y \geq 0\right\}$$

and

$$max\left\{b^Tu | A^Tu \leq c, F^Tu \leq f\right\}$$

- From the primal perspective, we obtain a relaxation
- A good restriction of the dual ployhedron is sought

Preprocessing and variable fixing

- Standard preprocessing techniques can be used in column generation approach
- Not directly on master problem, but in the solution space of the original formulation
- Bounds on subproblem variables can be tightened
- The reduced cost fixing techniques also permit to strengthen subproblem variables bounds further
- Can be used to estimate the range of dual price values

Initialization

RMP must be started with a feasible primal solution

- Construct heuristically/Introduce artificial columns
- Artifical variables penalized by a “big M” cost
- Smaller M gives a tighter upper bound on the respective dual variables (reduces *heading-in effect*)
- Combine phase 1 and 2, remaining artificial variables at the end of phase 2 can be dropped by increasing their cost
- Artificial columns are kept in the problem during branch-and-bound, useful to restart column generation after adding branching constraints
- They also stabilize the column generation procedure

Initialization

- Sometimes unit basis is already feasible, use estimate of actual cost coefficients instead of M
- After pre-processing phase, use estimates of the maximum and minimum value of master constraints to set artificial column coefficients
- Heuristic estimates of the optimal dual variable may be imposed as artificial upper bounds
- In primal, artificial columns act as slack variables and cuts in the dual. Thus they define upper bounds on dual prices
- Modifying their definitions dynamically is a way of controlling dual variables

Standard Column Generation

Several drawbacks from standard method for solving column generation

- a slow convergence (*tailing-off effect*)
- the first iterations produce irrevelant columns and dual bounds due to poor dual information(*heading-in effect*)
- degeneracy in the primal and hence multiple optimal solutions in the dual: results in master solution value remains constant for a while (*plateau effect*)
- instability in the dual solutions that are jumping from one extreme value to another (*bang-bang effect*)
- likewise, the intermediate Lagrangian dual bounds do not converge monotonically(*yo-yo effect*)

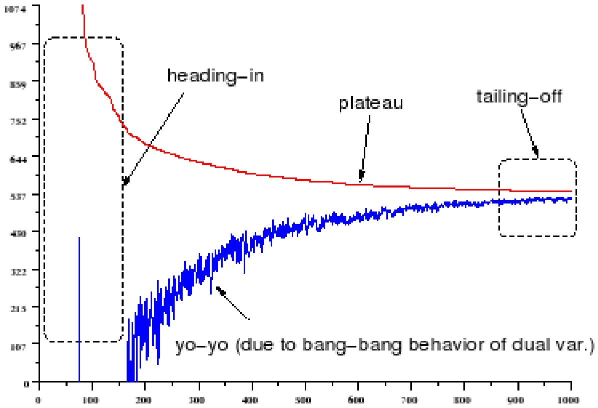

Figure 1: Illustration of the convergence of a simplex-based column generation approach on the TSP instance eil76 of the TSPLIB: the upper (resp. lower) curve gives the primal (resp. dual) bound at each iteration.

Stabilization techniques

Vignac classifies into 4 categories

- defining bounds on dual prices
- smoothing dual prices that are returned to subproblem
- penalizing deviation from a stability center (usually defined as the dual solution that gives the best Lagrangian bound value)
- working with interior dual solution rather than extreme dual solutions

Basic Stabilization techniques

Some of basic techniques that can be implemented within the Simplex method

- Re-optimization after adding a column rather than optimization from scratch (toward the previous primal solution)
- Penalizing deviation from a stability center (using L^1 norm)
- Better formulation of master problem
  - Avoid redundant constraint in the master(creates degeneracy)
  - Use inequality constraints rather than equality constraints(if they yields same solution)
  - Use aggregate formulation of master constraints, this may bring stability

Basic Stabilization techniques

- Managing artificial variables dynamically to control dual variables
- Intelligent initialization of artificial variables reduces heading-in effect
- Using a form of smoothing of dual prices
- Try to mimic the behaviour of alternative dual price updating methods (interior point, bundle method or ACCPM)
- In a branch-and-price algorithm, the exact solution of master LP might not always be needed, branch early or prune by bound

Pricing Problem

- We are free to any variable from non-basic variables
- Classical Dantzig rule chooses the column with the most negative reduced cost
- An approximation to pricing problem suffices until the last iteration
- Alternative pricing rules; *steepest-edge* pricing and *Devex* variant

Column re-optimization and post-processing

- Re-optimizing existing columns before calling on the oracle subproblem
- Apply a local search heuristic to active columns
- Introduce one column at a time and scan other after re-optimizing in multiple column generation procedure
- Use idea of lifting (Column generation lifting), increase column coefficients in master constraints by solving auxiliary subproblems

Column re-optimization and post-processing

- Use concept of *base-pattern*, It can be viewed as a seed for the generation of all columns that are represented by it
- To re-optimize a column, one can compute the best reduced cost column that can be obtained from its base-pattern
- For the cutting stock problem, a cutting pattern defines a partition of the wide-roll where cut pieces can be replaced by an assortment of smaller items
- An *exchange vector* is a valid perturbation: added to a feasible column, it gives another feasible column

The Tailing-off effect

- Simplex-based column generation is known as poor convergence, only little progress per iteration is made close to optimum
- It may be time consuming to prove optimality of a degenerate optimal solution
- Since primal and dual solutions are updated iteratively, very small adjustments may be necessary close to optimum
- A problem specific reformulation of the pricing problem may help, that restrict attention to a set of well structured columns
- One has to also cope with numerical instability

Bang-bang effect

Instability in the dual solutions that are jumping from one extreme value to another

- Simple idea is to bound the dual variable values
- Boxtep method creates a box around the current optimal solution to dual RMP. If the new solution is attained on the boundary, relocate the box
- Weighted Dantzig-Wolfe Decomposition uses a convex combination of last multiplier and best multipliers found so far (In the context of Lagrangian Dual)
- This idea can be combined with column generation process, i.e. every few iterations improve current dual solution by some subgradient algorithm iterations
- Trust Region Method can be used to define box constraints, trust region parameter is updated in each itearation depending on how well dual RMP approximates Lagrangian dual problem

Lower bounds and Early termination

- Stop generating columns when tailing off occurs and take a branching decision
- Assuming integral cost coefficients, column generation can be terminated as soon as $\lceil LB \rceil = \lceil \bar{z} \rceil$ ($\bar{z}$ is current feasible solution)
- With incumbent integral objective value $\hat{z}$, a node can be pruned as soon as $\lceil LB \rceil \geq \hat{z}$

Branching Decisions

- Compatible branching scheme is needed which prevents columns that have been set to zero on from being regenerated without a significant complication of pricing problem
- This would lead to finding the k^{th} best subproblem solution instead of optimal one
- Generally speaking, a decision imposed on the pricing problem is preferable to one imposed on the master problem as it directly controls the generated columns