Valid Inequalities for MILPs - III

(Split cuts and summary)

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References

- Giérard Cornuéjols, Valid Inequalities for Mixed Integer Linear Programs, manuscript.

Review

Gomory Mixed Integer Inequality

Consider:

$$S = \{(x,y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ | \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b\} \tag{1}$$

Substitute $a_j = \lfloor a_j \rfloor + f_j, b = \lfloor b \rfloor + f_0$.
Then,

$$\sum_{j=1}^{n} \lfloor a_j \rfloor x_j + \sum_{j=1}^{n} f_j x_j + \sum_{j=1}^{p} g_j y_j = \lfloor b \rfloor + f_0$$

$$\Rightarrow \sum_{f_j \leq f_0} \lfloor a_j \rfloor x_j + \sum_{f_j > f_0}^{n} (\lceil a_j \rceil - 1) x_j + \sum_{j=1}^{n} f_j x_j + \sum_{j=1}^{p} g_j y_j = \lfloor b \rfloor + f_0$$

$$\Rightarrow \sum_{f_j \leq f_0} \lfloor a_j \rfloor x_j + \sum_{f_j > f_0}^{n} \lceil a_j \rceil x_j + \sum_{f_j \leq f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j = \lfloor b \rfloor + f_0$$

$$\Rightarrow \sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j = \lfloor b \rfloor - \sum_{f_j \le f_0} \lfloor a_j \rfloor x_j - \sum_{f_j > f_0} \lceil a_j \rceil x_j + f_0$$

$$\Rightarrow \sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j = \lfloor b \rfloor - \sum_{f_j \le f_0} \lfloor a_j \rfloor x_j - \sum_{f_j > f_0} \lceil a_j \rceil x_j + f_0$$

For any (x, y) in S,

$$\sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j \le -1 + f_0$$

OR

$$\sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j \ge f_0$$

$$\Rightarrow \sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j = \lfloor b \rfloor - \sum_{f_j \le f_0} \lfloor a_j \rfloor x_j - \sum_{f_j > f_0}^{n} \lceil a_j \rceil x_j + f_0$$

$$\Rightarrow \sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j = \underbrace{\lfloor b \rfloor - \sum_{f_j \le f_0} \lfloor a_j \rfloor x_j - \sum_{f_j > f_0}^{n} \lceil a_j \rceil x_j}_{K} + f_0$$

For any (x, y) in S,

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$$\Rightarrow \sum_{f_j \leq f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j = \underbrace{\lfloor b \rfloor - \sum_{f_j \leq f_0} \lfloor a_j \rfloor x_j - \sum_{f_j > f_0}^{n} \lceil a_j \rceil x_j}_{K} + f_0$$

For any (x, y) in S,

$$\sum_{f_j \leq f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j \leq -1 + f_0$$

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$$-\sum_{f_j \le f_0} f_j x_j + \sum_{f_j > f_0} (1 - f_j) x_j - \sum_{j=1}^{p} g_j y_j \ge 1 - f_0$$

OR

$$\sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j \ge f_0$$

$$\sum_{f_j \le f_0} \frac{f_j}{1 - f_0} x_j - \sum_{f_j > f_0} \frac{(1 - f_j)}{1 - f_0} x_j + \sum_{j=1}^{p} \frac{g_j}{1 - f_0} y_j \ge 1$$

OR

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{(1 - f_j)}{f_0} x_j + \sum_{j=1}^{p} \frac{g_j}{f_0} y_j \ge 1$$

$$-\sum_{f_j \le f_0} f_j x_j + \sum_{f_j > f_0} (1 - f_j) x_j - \sum_{j=1}^{p} g_j y_j \ge 1 - f_0$$

OR

$$\sum_{f_j \le f_0} f_j x_j - \sum_{f_j > f_0} (1 - f_j) x_j + \sum_{j=1}^{p} g_j y_j \ge f_0$$

$$\sum_{f_j \le f_0} \frac{f_j}{1 - f_0} x_j - \sum_{f_j > f_0} \frac{(1 - f_j)}{1 - f_0} x_j + \sum_{j=1}^{p} \frac{g_j}{1 - f_0} y_j \ge 1$$

OR

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{(1 - f_j)}{f_0} x_j + \sum_{j=1}^{p} \frac{g_j}{f_0} y_j \ge 1$$

K-cuts, Reduce and Split Cuts

1. Any equality which is valid for S can be used to generate a GMI.
2. Simplex Tableau gives an equality straightaway.
3. What if we want more such equalities?
4. Take linear combinations of two/more equalities: Reduce and Split Cuts.
5. Just divide a single equation by K: K-cut

K-cuts

For a given equality constraint, a GMI inequality is:

$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{(1-f_j)}{1-f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j \le 0} \frac{g_j}{1-f_0} \ge 1$$

Observation:

- Regardless of a_j, coefficient of x_j in above $\in [0, 1]$.
- Multiplying by $K > 1$ increases the coefficients of y
- Keep a_j as integer, reduce g_j.
- Also see for equivalence with t-MIR cuts: Sanjeeb Dash and Oktay Günlük. Valid inequalities based on simplex mixed-integer sets. *Mathematical Programming*, 105:29–53, 2006. Series A.
- This may weaken the inequality. Use small values of K.

Reduce and Split Cuts

Intuition: Keep coefficients of y low.

- Use *Basis Reduction*!
- Gives Striking results on some problems. vpm2: GMI–38K nodes, R&S–4K nodes.
- Ineffective on others.

MIR–inequalities

Consider the set S:

$$S = \{(x, y) \in \mathbb{Z} \times \mathbb{R}_+ | x - y \leq b$$

Proof:- Consider two cases $x \leq \lfloor b \rfloor$ and $x \geq \lfloor b \rfloor + 1 \ldots$

- Note: This is slightly different from MIR-inequalities in Nemhauser and Wolsey.
- In N&W, let $T = \{x \in \mathbb{Z}^n_+, y \in \mathbb{R}^p_+ : Ax + Gy \leq b\}$.
- Many heuristics: Dash, Wolsey, …

Split–inequalities

Let $P = \{(x, y) \in \mathbb{R}^{n+p} | Ax + Gy \leq b\}$. Let $S = P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. Let $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$.
Suppose,

$$\Pi_1 = P \cap \{(x, y) | \pi x \leq \pi_0\}$$
$$\Pi_2 = P \cap \{(x, y) | \pi x \geq \pi_0 + 1\}$$

Then, if $cx + hx \geq c_0$ is a valid inequality for Pi_1, Pi_2, then it is valid for S.

Split Cuts are everything

Following are split cuts:

- GMI
- K-cuts, reduce-and-split cuts
- MIR
- Lift and Project (and their strengthening)

Split Closure

- Chvátal closures for mixed integer programming problems. *Mathematical Programing*, 47:155–174, 1990.
- Split Closure is a polyhedron
- For Mixed Binary Programs, $n-$th split closure is sufficient.
- It is also necessary.
- Balas and Saxena show that Split Closure can be very tight.

Split Closure is NOT enough ...

... for MIPs

But there are alternatives . . .