A Lagrangean based Branch-and-Cut algorithm for global optimization of MINLP with decomposable structures

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MINLP Formulation

$$
\begin{aligned}
z = \text{minimize} \quad & s(x,y) + \sum_{n=1}^{N} r_n(u_n, v_n) \\
\text{subject to} \quad & h_n(u_n, v_n) = 0, \quad n = 1, \ldots, N, \\
& g_n(u_n, v_n) \leq 0, \quad n = 1, \ldots, N \\
& h_n^{'}(x, y, u_n, v_n) = 0, \quad n = 1, \ldots, N \\
& g_n^{'}(x, y, u_n, v_n) \leq 0, \quad n = 1, \ldots, N \\
& x^L \leq x \leq x^U \\
& y \in \{0,1\}^J \\
& u_n^L \leq u_n \leq u_n^U, \quad n = 1, \ldots, N \\
& v_n \in \{0,1\}^{m_{v_n}}, \quad n = 1, \ldots, N \\
& x \in \mathbb{R}^I, u_n \in \mathbb{R}^{m_{u_n}}
\end{aligned}
\tag{P}
$$

Model Reformulation

- Linking variables are x, and y and constraints are $h_n^{'}()$ and $g_n^{'}()$.
- Contraints $h_n()$, $g_n()$, $h_n^{'}()$ and $g_n^{'}()$ may be nonconvex.
- To decompose P, create identical copies of x and y.
- Given by $\{x^1, x^2, \ldots, x^N\}$ and $\{y^1, y^2, \ldots, y^N\}$.
- Linking variables are the same across all sub-models (non-anticipativity).
- $x_1 = x_2 = \ldots = x^N$
- $y_1 = y_2 = \ldots = y^N$
- Can be expressed in the model as:
- $x^n - x^{n+1} = 0 \quad n = 1, \ldots, N-1.$
- $y^n - y^{n+1} = 0 \quad n = 1, \ldots, N-1.$

$$
\begin{aligned}
z^{\text{RP}} = \text{minimize} \quad & \sum_{n=1}^{N} w_n s(x^n, y^n) + \sum_{n=1}^{N} r_n(u_n, v_n) \\
\text{subject to} \quad & h_n(u_n, v_n) = 0, \quad n = 1, \ldots, N, \\
& g_n(u_n, v_n) \le 0, \quad n = 1, \ldots, N \\
& h_n^{'}(x^n, y^n, u_n, v_n) = 0, \quad n = 1, \ldots, N \\
& g_n^{'}(x^n, y^n, u_n, v_n) \le 0, \quad n = 1, \ldots, N \\
& x^n - x^{n+1} = 0, \quad n = 1, \ldots, N-1 \\
& y^n - y^{n+1} = 0, \quad n = 1, \ldots, N-1 \\
& x^L \le x^n \le x^U, \qquad y^n \in \{0, 1\}^J \\
& u_n^L \le u_n \le u_n^U, \quad n = 1, \ldots, N \\
& v_n \in \{0, 1\}^{m_{v_n}}, \quad n = 1, \ldots, N \\
& x^n \in \mathbb{R}^I, u_n \in \mathbb{R}^{m_{u_n}}
\end{aligned} \tag{RP}
$$

Solution Methodology

General approach to globally optimize P:

- Do branch and bound
- Solve relaxations constructed by convexifying the nonconvex terms.
- Relaxations are generally weak.

Insight: Model is decomposable, can be used to derive tight bounds.

- Use Lagrangean decomposition to decompose P into N' sub-problems.
- Use the solution of the sub-problems to obtain relaxation strengthing cuts.

Solution Methodology...

- Use branch-and-cut framework and solve problems at every node to global optimality.
- At a node, solve convex relaxation of original nonconvex model with added cuts.
- $\Rightarrow$ tighter lower bounds.
- Obtain upper bounds by fixing binary variables and solving the nonconvex NLP to global optimality.

Lagrangean Relaxation:

$$
\begin{aligned}
z^{\text{LRP}} = \text{minimize} \quad & \sum_{n=1}^{N} w_n s(x^n, y^n) + \sum_{n=1}^{N} r_n(u_n, v_n) + \\
& \sum_{n=1}^{N-1} (\bar{\lambda}_n^{x})(x^n - x^{n+1}) + \sum_{n=1}^{N-1} (\bar{\lambda}_n^{y})(y^n - y^{n+1}) \\
\text{subject to} \quad & h_n(u_n, v_n) = 0, \quad n = 1, \ldots, N, \\
& g_n(u_n, v_n) \leq 0, \quad n = 1, \ldots, N \\
& h_n^{'}(x^n, y^n, u_n, v_n) = 0, \quad n = 1, \ldots, N \\
& g_n^{'}(x^n, y^n, u_n, v_n) \leq 0, \quad n = 1, \ldots, N \\
& x^L \leq x^n \leq x^U, \qquad y^n \in \{0, 1\}^J \\
& u_n^L \leq u_n \leq u_n^U, \quad n = 1, \ldots, N \\
& v_n \in \{0, 1\}^{m_{v_n}}, \quad n = 1, \ldots, N \\
& x^n \in \mathbb{R}^I, u_n \in \mathbb{R}^{m_{u_n}}
\end{aligned}
\tag{LRP}
$$

Decompose LRP into sub-problems SP_n, $n = 1, \ldots, N$:

$$
\begin{aligned}
z_n = \text{minimize} \quad & w_n s(x^n, y^n) + r_n(u_n, v_n) + \\
& (\bar{\lambda}_n^x - \bar{\lambda}_{n-1}^x)(x^n) + (\bar{\lambda}_n^y - \bar{\lambda}_{n-1}^y)(y^n) \\
\text{subject to} \quad & h_n(u_n, v_n) = 0 \\
& g_n(u_n, v_n) \leq 0 \\
& h_n^{'}(x^n, y^n, u_n, v_n) = 0 \\
& g_n^{'}(x^n, y^n, u_n, v_n) \leq 0 \\
& x^L \leq x^n \leq x^U, \qquad y^n \in \{0,1\}^J \\
& u_n^L \leq u_n \leq u_n^U \\
& v_n \in \{0,1\}^{m_{v_n}} \\
& x^n \in \mathbb{R}^I, u_n \in \mathbb{R}^{m_{u_n}}
\end{aligned}
\qquad (SP_n)
$$

Using solution of Lagrangean sub-problems

- Solve each sub-problem SP_n to global optimality for fixed λ.
- $\sum_{n=1}^{N} z_n^* = z^{LB}$ is a valid lower bound for P.
- Tightest possible lower bound obtained from the solution of the lagrangean dual:
- $z^D = \max_{\bar{\lambda}} z^{LB}$.
- Hard problem to solve...
- Instead, use a heuristic by Fisher(1981) to iterate with different values of Lagrange multipliers. Multiplier updating rules...
- Therefore, use $\sum_{n=1}^{N} z_n^{L^*} = z^{LB}$, where L^* is the highest valued lower bound on the global optimum of SP_n.

Using solution of Lagrangean sub-problems...

Optimality based cutting planes:

- Let z_n^* be the global optimum for SP_n.

$$z_n^* \leq w_n s(x,y) + r_n(u_n, v_n) + (\bar{\lambda_n}^x - \bar{\lambda_{n-1}}^x)(x) + (\bar{\lambda_n}^y - \bar{\lambda_{n-1}}^y)(y) \qquad (C_n)$$

Theorem

The cuts C_n, $n = 1, \ldots, N$ are valid for RP, and therefore for P.

- In practice, z_n^* is replaced by $z_n^{L^*}$ in C_n.
- C_n is added to P.

$$
\begin{aligned}
z^{P'} = \text{minimize} \quad & s(x,y) + \sum_{n=1}^{N} r_n(u_n, v_n) \\
\text{subject to} \quad & h_n(u_n, v_n) = 0, \quad n = 1, \ldots, N, \\
& g_n(u_n, v_n) \leq 0, \quad n = 1, \ldots, N \\
& h'_n(x, y, u_n, v_n) = 0, \quad n = 1, \ldots, N \\
& g'_n(x, y, u_n, v_n) \leq 0, \quad n = 1, \ldots, N \\
& z_n^* \leq w_n s(x,y) + r_n(u_n, v_n) + \\
& (\bar{\lambda}_n^x - \bar{\lambda}_{n-1}^x)(x) + (\bar{\lambda}_n^y - \bar{\lambda}_{n-1}^y)(y), \quad n = 1, \ldots, N \\
& x^L \leq x \leq x^U, \qquad y \in \{0,1\}^J \\
& u_n^L \leq u_n \leq u_n^U, \quad n = 1, \ldots, N \\
& v_n \in \{0,1\}^{m_{v_n}}, \quad n = 1, \ldots, N \\
& x \in \mathbb{R}^I, u_n \in \mathbb{R}^{m_{u_n}}
\end{aligned}
\qquad (P')
$$

convexify constraints.. $\Rightarrow$ convex relaxation of P':

$$
\begin{aligned}
z^R = \text{minimize} \quad & \bar{s}(x,y) + \sum_{n=1}^{N} \bar{r}_n(u_n, v_n) \\
\text{subject to} \quad & \bar{h}_n(u_n, v_n) = 0, \quad n = 1, \ldots, N, \qquad (R) \\
& \bar{g}_n(u_n, v_n) \le 0, \quad n = 1, \ldots, N \\
& \bar{h}'_n(x, y, u_n, v_n) = 0, \quad n = 1, \ldots, N \\
& \bar{g}'_n(x, y, u_n, v_n) \le 0, \quad n = 1, \ldots, N \\
& z_n^* \le w_n \bar{s}(x,y) + \bar{r}_n(u_n, v_n) + \\
& (\bar{\lambda}_n^x - \bar{\lambda}_{n-1}^x)(x) + (\bar{\lambda}_n^y - \bar{\lambda}_{n-1}^y)(y), \quad n = 1, \ldots, N \\
& x^L \le x \le x^U, \qquad y \in \{0,1\}^J \\
& u_n^L \le u_n \le u_n^U, \quad n = 1, \ldots, N \\
& v_n \in \{0,1\}^{m_{v_n}}, \quad n = 1, \ldots, N \\
& x \in \mathbb{R}^I, u_n \in \mathbb{R}^{m_{u_n}}
\end{aligned}
$$

- For specific nonconvex terms, special convex estimators exist (Tawarmalani and Sahinidis, 2002).
- Solve R to obtain lower bounds.

Theorem

The lower bound obtained by solving R is at least as strong as the one obtained by solving a convex relaxation of P obtained by convexifying the nonconvex terms.

- Obtaining lower bound by this procedure is computationally expensive.
- But this reduces the number of nodes in the tree search significantly...
- Can decompose to $N' < N$ sub-problems
- Branching in the algorithm is done on the linking variables x and y.
- CONOPT 3.0 and BARON 7.2.5 used for solving NLP problems.
- CPLEX 9.0 used for solving LP and MILP problems.
- DICOPT and BARON 7.2.5 used for solving MINLP problems.