A review of François Margot’s paper

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Preliminaries

- Let \( \Pi^n \) be the set of all permutations of \( I^n = \{1, \ldots, n\} \).
- \( \Pi^n \) is the symmetric group of \( I^n \).
- \( \pi \in \Pi \) is an n-vector.
- \( \pi[i] \) is the image of \( i \) under \( \pi \).
- Let \( w \) be the vector obtained by permuting \( v \) according to \( \pi \):
  \[
  w[\pi[i]] = v[i] \quad \text{for all } i \in I^n
  \]
Preliminaries

- Consider the following ILP

\[
\begin{align*}
\text{min } & \quad c^T x \\
\text{s.t. } & \quad Ax \geq b, \\
x & \in \{0, 1\}^n,
\end{align*}
\]

- WLOG we can assume \( A, b, \) and \( c \) are all integers

- Let \( \pi \) be a permutation of \( n \) variable, \( \sigma \) a permutation

- Let \( A(\pi, \sigma) \) be the matrix obtained from \( A \) by permuting variables and columns

- Let \( G = \{ \pi | \pi(c) = c \text{ and } \exists \sigma \text{ s.t. } \sigma(b) = b, A(\pi, \sigma) = A \} \)
Preliminaries

- **Definition**: The orbit of $S$ under $G$ is

\[ \text{orb}(S, G) = \{ S' \subseteq I^n | S' = g(S) \text{ for some } g \in G \} \]

- **Definition**: the stabilizer of $S$ in $G$ is:

\[ \text{stab}(S, G) = \{ g \in G | g(S) = S \} \]

- Denote $F_k^a$ to be the set of variables fixed to $k$ at node $a$.

- $N^a$ the set of variables not fixed at node $a$.
• Subproblems are isomorphic if $\exists$ a permutation $g \in G$ with $g(F^a_k) = F^b_k$ for $k = 0, 1$.

• Using this definition is difficult.
  ✴ How do you find $g$ for given nodes $a$ and $b$?
  ✴ This will have to be done a lot.
Ranked Branching Rule

- Goal: evaluate a single node, not pairs of nodes
- Let R be a rank vector, indicating the order in which variables have been used for branching
- The rule to select the branching variable $x_f$ at a is:
  
  \((i)\) If $\exists j \in N^a$ with $R[j] < n + 1$, then $f = \text{arg min} \{ R[j] \mid j \in N^a \}$

  \((ii)\) Else, choose $f \in N^a$
Ranked Branching Rule

- Let $J = \{j_1, \ldots, j_p\}$ be unordered multiset of $I^{n+1}$
- Let $J^*$ be the ordered multiset formed by listing $J$ in non-decreasing order
- Given set $J_i$, $J_j$, $J_i \preceq J_j$ if $J_i$ is lexicographically smaller
- For a given $R$, $J$ is a representative of the sets in its orbit if $J$ is lex. min. under $G$:
  \[ R(j) \preceq R(g(J)) \quad \forall g \in G \]
Lemma 1

- Let $R_1$ and $R_2$ be two rank vectors obtained during branch and cut, assume $R_2$ is obtained after $R_1$, then:

- (i) If $J$ is not a representative w.r.t. $R_1$, then $J$ is not a representative w.r.t. $R_2$

- (ii) If $J$ is a rep. w.r.t. $R_1$ and all entries in $R(j)$ are less than $n+1$, then $J$ is the unique rep. to $R_1$

- (iii) If $J$ is a rep w.r.t. $R_1$ and all entries in $R(j)$ are less than $n+1$, then $J$ is a rep w.r.t. $R_2$
Lemma 2

Let \( J \subseteq I^n \) be a rep under G w.r.t. R. Let \( j' := J - j \) with \( j \in \text{arg max}\{R[i]|i \in J\} \). Then \( J' \) is also a rep w.r.t. R.

- Isomorphism Pruning: If \( F_1^a \) is not a representative, prune node a.
• At node a, all variables in the orbit of \( \text{stab}(F_1^a) \) can be set to k as soon as we know any variable can be set to k.

• Standard setting algorithms let you set a variable to k when you can show \( \exists \) an optimal solution with that variable equal to k. That does not work, that solution can be pruned by isomorphic pruning.

• If you are able to set \( x_i = k \), then for any \( g \in G \), we can set \( x_{g(i)} \) to k.