Integer Programming Duality

CORAL Seminar Series

Scott DeNegre

17 March 2005

References

- Gomory, R.E., An algorithm for integer solutions to linear programs. In R.L. Graves and P.Wolfe, editors, *Recent Advances in Mathematical Programming*, 269-302. McGraw-Hill, New York, 1963.
- Klamroth, K., J. Tind, and S. Zust. Integer programming duality in multiple objective programming. *Journal of Global Optimization*,29 , 1-18 , 2004.
- Nemhauser G.L. and L.A. Wolsey, *Integer and Combinatorial Optimization*. John Wiley & Sons, Inc., New York, 1999.
- Wolsey, L.A., Integer programming duality: Price functions and sensitivity analysis. *Mathematical Programming*, 20, 173-195, 1981.

Basic Definitions

- Let $\mathcal{F}$ be the set of nondecreasing functions $F : \mathbb{R}^m \rightarrow \mathbb{R}$

$$\mathcal{F} := \{(F : \mathbb{R}^m \rightarrow \mathbb{R}) : F(a) \leq F(b) \forall a, b \in \mathbb{R}^m, a \leq b\}$$

- A function F is said to *superadditive* if $F(a) + F(b) \leq F(a+b) \forall a, b \in \mathbb{R}^m$.

- Let $\mathcal{G}$ be the set of nondecreasing superadditive functions

$$\begin{aligned} \mathcal{G} := \{(F : \mathbb{R}^m \rightarrow \mathbb{R}) \quad : \quad & F(0) = 0, (F(a) \leq F(b) \forall a, b \in \mathbb{R}^m, a \leq b), \\ & \text{and } (F(a) + F(b) \leq F(a+b) \forall a, b \in \mathbb{R}^m)\} \end{aligned}$$

Generalized Dual

- Consider the IP

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t.} \quad & Ax \leq b \qquad (1) \\
& x \geq 0, \ \text{integer}
\end{aligned}
$$

- We can write the dual of (1) as

$$
\begin{aligned}
\min \quad & F(b) \\
\text{s.t.} \quad & F(A_j) \geq c^t \quad j = 1, \ldots, n \qquad (2) \\
& F \in \mathcal{G}
\end{aligned}
$$

  – When we consider generation of F via specific IP algorithms, we will see a generalization of this dual that does not require superadditivity.

Main IP Duality Results

Theorem 1. [Weak Duality] *$c^T x \leq F(b)$ for all feasible solutions x of (1) and all dual feasible functions F of (2).*

Theorem 2. [Strong Duality] *If either (1) or (2) has a finite optimal solution, then there exists solutions x^* of (1) and F^* of (2) such that $c^T x^* = F^*(b)$. Further if (1) is infeasible, then (2) is either infeasible or unbounded, and if (2) is infeasible, then (1) is either infeasible or unbounded.*

Theorem 3. [Complementary Slackness] *Let x^*, F^* be optimal solutions to (1) and (2), respectively. Let $s^* = b - Ax^*$ and $v_j^* = c_j - F^*(A_j), j = 1 \ldots, n$, then*

- $v_j^* \leq 0, j = 1 \ldots, n$
- *if $x_j^* > 0$, then $v_j^* = 0$*
- $F^* \in \mathcal{G}$
- *$F^*(b) = F^*(Ax^*)$ and $F^*(s^*) = 0$.*

Sensitivity Analysis

Let $x^*, F^* \in \mathcal{G}$ be an optimal solution pair with associated optimal value Z^* and let x' and Z' denote the optimal solution and value, respectively of the problem after a change in the original problem.

- $b \to b'$
  - F^* remains dual feasible $\Rightarrow Z' \leq F^*(b')$
  - If F^* is still optimal, then $x' \in Y^* := \{y | F^*(Ay) = c^T y\}$
- $c \to c'$
  - x^* remains feasible $\Rightarrow Z' \geq (c')^T x^*$
  - If $c'_j \leq F^*(A_j) \forall j$, F^* remains feasible $\Rightarrow Z' \leq F^*(b)$
  - If $c'_j \leq F^*(A_j)$ when $x^*_j = 0$ and $c'_j = c_j$ when $x^*_j > 0$, x^* remains optimal

Sensitivity Analysis

- New variable added, $(\bar{c}, \bar{A})$
  - x^* remains primal feasible $\Rightarrow Z' \geq Z$
  - x^* remains optimal if $F^*(\bar{A}) \geq \bar{c}$
- New constraint added $a_0' x \leq b_0$
  - If x^* is still feasible, it is optimal
  - $\bar{F} : \mathbb{R}^{m+1} \to \mathbb{R}$ defined by $\bar{F}(d, d_{m+1}) = F^*(d)$ is dual feasible for the new problem $\Rightarrow Z' \leq Z = \bar{F}(b, b_0)$

Generation of Dual Optimal Functions

We will focus on two methods for solving IPs

- Cutting Planes
- Branch and Bound

Cutting Plane Algorithm

In iteration $r \geq 0$ of the algorithm, we solve an LP (P_r) of the form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m + r \\ & x \geq 0 \end{aligned} \tag{3}$$

- If (P_r) is infeasible or x^* integer, we are done.
- Otherwise, $\exists\ a^{m+r+1} = (a_{m+r+1,1}, \ldots, a_{m+r+1,n}) \in \mathbb{R}^n$ and $b_{m+r+1} \in \mathbb{R}$ such that

  $$a^{m+r+1} x = b_{m+r+1}$$

  is a *separating hyperplane*.

Cutting Plane Algorithm

Specifically, we have

$$a_{m+r+1,j} = G^{r+1}(A_j), \quad b_{m+r+1} = G^{r+1}(b)$$

where

$$\begin{aligned} G^{r+1}(d) &:= \left\lfloor \sum_{i=1}^{m} \lambda_i^r d_i + \sum_{i=1}^{r} \lambda_{m+i}^r G^i(d) \right\rfloor \\ \lambda^r &= (\lambda_1^r, \ldots, \lambda_{m+r}^r) \geq 0 \end{aligned}$$

The cut $a^{m+r+1}x \leq b_{m+r+1}$ is added as the $(m+r+1)^{\mathrm{st}}$ constraint

- See Chvatal (1993) or Nemhauser and Wolsey (1999) for the details.

Dual Function Construction

Proposition 1. *Let $u^r \in \mathbb{R}_+^{m+r}$ be a dual feasible solution of (P_r). Then the functions*

$$F^r(d) := \sum_{i=1}^{m} u_i^r d_i + \sum_{i=1}^{r} u_{m+i}^r G^i(d) \tag{4}$$

are superadditive dual feasible functions for (1). If x^r is optimal for (P_r) and u^r is dual optimal for (P_r), $c^T x^r = F^r(b)$.

This leads to a constructive version of Theorem 2.

Theorem 4. *Suppose the cutting plane algorithm terminates finitely when applied to (1).*

- *If (1) has a finite optimal solution, then $\exists$ an optimal feasible solution x^r of (1) and a dual optimal function F^r of (2) of the form (4) such that $c^T x^r = F^r(b)$.*
- *If (1) is infeasible, then $\exists$ a dual function F^r of the form (4) satisfying $F^r(A_j) \geq 0, j = 1, \ldots, n$ and $F^r(b) < 0$, and (2) is unbounded.*

Alternate Dual Formulation

In general, Branch and Bound will not produce superadditive functions. This leads the introduction of a more general dual of (1)

$$\begin{aligned} \min \quad & F(b) \\ \text{s.t.} \quad & F(Ax) \geq c^t x \quad \forall x \geq 0, \text{ integer} \\ & F \in \mathcal{F} \end{aligned} \tag{5}$$

It is easily shown that (5) is a strong dual for (2).

Branch and Bound

In B & B, we replace the original problem (1) with a finite series of subproblems (P_t)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \qquad (6) \\ & x \in X_t \qquad (7) \end{aligned}$$

for $t = 1, \ldots, r$ such that $\{x \in \mathbb{R}^n : x \geq 0, \ \text{integer}\} \subseteq \bigcup_{t=1}^{r} X_t$.

- The algorithm terminates if
  - All subproblems are infeasible
  - A solution $x^{t^*} \in \mathbb{Z}$ is optimal for (P_{t^*}) such that $c^T x^{t^*} = z_t^* \geq z^t$ for all $t \neq t^*$
- Otherwise, the algorithm continues with further division of at least one subproblem.

Dual Function Construction

Proposition 2. *If* $F_t \in \mathcal{F}$ *are dual feasible functions for* (P_t) *(i.e.* $F_t(Ax) \geq c^T x \forall x \in X_t, t = 1 \ldots, r$*) then*

$$F(d) := \max_{t=1,\ldots,r} F_t(d)$$

is a dual feasible function for (1).

Under some mild assumptions (which will hold using LP-based branch and bound), we have

Theorem 5. *Let (1) have finite optimum. If a linear programming based branch and bound algorithm terminates with a finite series of subproblems* $(P_t), t = 1, \ldots, r$*, then* $\exists$ *a dual optimal function* $F \in \mathcal{F}$ *of the form*

$$F(d) := \max_{t=1,\ldots,r} (\pi^t d + \alpha^t), \quad \alpha^t \in \mathbb{R}, \pi^t \in \mathbb{R}^m, \pi^t \geq 0. \tag{8}$$