CLOAKING WITH THE DAMPED WAVE EQUATION

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BACHELOR OF SCIENCE

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Bachelor of Science.

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Bachelor of Science.

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(Dr. XXX)

Approved by the Department of Mathematics.

Preface

It is fascinating to work on a topic related to cloaking, especially when the central idea is intuitive and straightforward, but not that well-researched from the perspective that I take. The important parts of this thesis are supported by a certain level of theoretical analysis, but many of them could be improved and generalized.

This thesis is the first relatively long-term mathematical research I have done, and I wish it is going to be a good start for my Ph.D. study.

Acknowledgements

I would like to thank Dr. Bryan for being my advisor. He introduced me to a fascinating topic that I did not think of and guided me throughout the entire year. Some enlightenment provided by him is essential to this thesis.

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Chapter 1

Introduction

In this thesis we examine the mathematical feasibility of cloaking with the wave equation. Traditional approaches to cloaking attempt to "wrap" an object inside a layer of special material in order to hide it from an outside observer. See [3] for an example in the context of impedance imaging. We instead seek to cloak the outside world from an observer, by enclosing him inside a region bounded by a finite thickness wall coated with a special material. It is assumed the observer uses emitted and reflected waves—like radar—in an attempt to image the exterior of the region. We consider the possibility of using some method to prevent the outgoing waves from reflecting off this wall. As a result, the observer thinks the waves have traveled away to "infinity," and is unaware of the presence of the outside world. In reality, the perfect prevention of reflections may not be required, but the amplitude of the reflected wave should be lowered below the minimum sensitivity of the wave detector.

There are two approaches one might consider: constructing special boundary conditions at the wall, or adding some special medium (i.e. *buffer area*) in the path of the wave propagation. One of the studies in the first direction was done by Velasco et al., where they constructed transparent boundary conditions based on the D′Alembert solution and spinor-like formalism [10].

As shown in Figure 1.1, the buffer area in the second approach aims to reduce the amplitude or the energy of the wave so that when it is reflected and received by the observer, the amplitude or the energy of the wave is 0. Thus, no reflection will be detected.

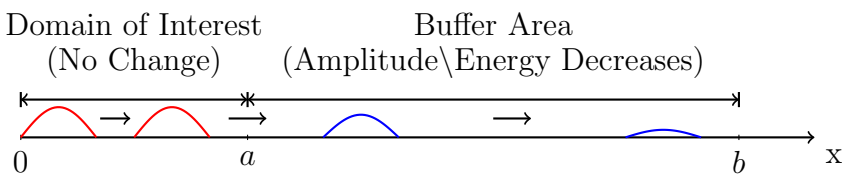


Figure 1.1: Demonstration for "buffer region" that absorbs wave energy to prevent reflection. When the wave propagates in $0 < x < a$ it is not attenuated. However, when the wave move into the buffer area $x > b$, its amplitude or energy decreases rapidly; very little energy is left after the wave reflects of a hard boundary at $x = b$ and returns to the observer in $(0, a)$.

Related to the second approach, many studies have looked into the behavior of the wave equation with different terms and coefficients, such as a damping term. For the one-dimensional wave equation with damping term, Cox and Zuazua proved the solution has a uniformly exponential energy decay rate [4] and, later, Freitas and Zuazua completed the proof of the stability of the solution in the previous work [5].

In physics, a method called a perfectly matched layer (PML) also has the idea of the second approach but mathematically achieves it through an entirely different process. It was firstly introduced by Bérenger [2] to truncate the space domain in Maxwell's equations. We present more on PML and provide the PML formulation for our problem of interest in Chapter 3.

In Chapter 4, we conduct a numerical experiment to demonstrate both our method and PML formulation along with some other interesting results.

1.1 Problem Statement

We now formulate the problem described above into that of solving a boundary value problem for the wave equation. First, consider the wave equation in the half-line $x > 0$ with prescribed initial data and data at $x = 0$, governed by

$$u_{tt} - c^2 u_{xx} = 0 \tag{1.1}$$
$$\begin{cases} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= 0 \end{cases}.$$

In this case right moving waves propagate to infinity without reflection. If an observer is present near $x = 0$, say in the interval $0 < x < a$ as in Figure 1.1, and initiates a right moving solution by choosing f and g appropriately, the resulting wave will not return.

Now consider a related problem on a bounded domain $0 < x < b$ where $b >> a$ is large. Let $\mu(x) \geq 0$ be a damping coefficient supported in $x > a$, and consider the problem

$$u_{tt} - c^2 u_{xx} + \mu(x) u_t = 0 \tag{1.2}$$
$$\begin{cases} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= 0 \\ u(b,t) &= 0 \end{cases}, \tag{1.3}$$

Here μ is a frictional term that damps u in proportion to velocity. We want to choose μ and b so that the solution of the boundary value problem (1.2) is the same as the solution of an initial value problem (1.1) in the region $0 < x < a$ and $t > 0$, or at least arbitrarily close. We don't want the observer in $(0, a)$ to detect the wall at $x = b$, or even deduce that there is damping present for $x > a$. The difficulty is that the transition from $\mu = 0$ to $\mu > 0$ at $x = a$ itself can induce reflections (thus making the damping detectable) and if not done properly, reflected waves will return from $x = b$ as well.

For simplicity we set $a = 1$ in what follows, and we only consider the waves traveling from left to right. Also, damping out reflected waves requires that the right end of the interval be rather far away at $x = b$ with $b \approx 50$. Since such a thick layer of damping material seems impractical, after designing a suitable damping region we consider a transformation of the interval $(1, b)$ to a much smaller region $(1, 1+\epsilon)$ where $\epsilon \approx 0$. Under such a change of variable the wave equation changes form slightly, but the transformed equation has altered coefficients that have a physical interpretation.

We thus obtain a "damping" later of only thickness ϵ with certain physical properties. We detail this change of variable in the next Chapter.

Chapter 2

Methodology

Our method for lowering and eliminating the amplitude of the reflected wave is simple: adding a properly chosen damping term to the wave equation, which corresponds to a layer of homogeneous or non-homogeneous materials that attenuates waves.

2.1 General Description

Initially, the damping is zero in the spatial region $0 < x < 1$, then the damping coefficient becomes positive in the buffer area $x > 1$. In this case, the solution to the modified wave equation (1.2) in $0 < x < 1$ has a solution that identical to that of the undamped equation on $0 < x < 1$, for as waves propagate to the right in the damped equation they are attenuated rapidly, with little reflection from the right end of the interval.

In what follows we assume the right end of the damped equation domain is at $x = 1 + \delta$ for some $\delta > 0$ (so $b = 1 + \delta$ in the previous notation); note δ is not small. We refer the part of the space domain $1 < x \leq 1 + \delta$ as the *buffer area*. Mathematically, $\mu(x)$ will take the form

$$\mu(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ \mu_0(x) & 1 < x \leq 1 + \delta \end{cases}, \tag{2.1}$$

where $\mu_0 > 0$ for all $x > 1$.

2.2 Form of the Solution and Decay Rate

Cox and Zuazua in [4] proved the solution of (1.2) has energy decay rate $Ce^{-h(x)t}$ where C is a positive constant and $0 < \mu(x) \leq h(x)$ for all x. By assuming the form of

the solution of (1.2), we can also bound the amplitude of the wave. What follows is not a rigorous proof of how the solutions to the damped wave equation decay, but a heuristic argument (but one which is well-supported by simulations).

We first write (1.2) as a system of first-order equations by adding an auxiliary function $z(x,t)$:

$$\begin{cases} u_t &= cz_x - \mu(x)u \\ z_t &= cu_x \end{cases}. \tag{2.2}$$

Consider the case when $u(x,t)$ is a right-traveling wave, for $x \in [0,1]$ and there is no damping in the solution, so we assume $u(x,t) = f(x-ct)$ where $f(x)$ is the initial wave in (1.3). We suppose f is supported in the interval $(0,1)$. Then $u(x,t)$ has initial speed $-cf'(x)$, i.e. $g(x) = -cf'(x)$ in (1.3).

By using a suitable numerical (finite difference) method to solve the (1.1) with a few examples, we observe that the amplitude of a right-moving wave decreases before it reaches the boundary at $x = 1+\delta$. Thus, we guess $u(x,t)$ has the form $a(x)f(x-ct)$ and $a(x) = 1$ for $x \in [0,1]$. By using $u_t = cz_x$ with $\mu = 0$ in (2.2) we deduce that $-cf'(x) = cz_x(x,0)$ so that

$$z(x,0) = -f(x) \tag{2.3}$$

Similarly, we guess $z(x,t)$ has the form $b(x)f(x-ct)$ and $b(x) = 1$ for $x \in [0,1]$.

However, for $1 < x < 1+\delta$ we expect $a(x)$ and $b(x)$ to decrease. Here (2.2) can be written as

$$-ca(x)f'(x-ct) = c[b'(x)f(x-ct) + b(x)f'(x-ct)] - \mu(x)a(x)f(x-ct) \tag{2.4}$$
$$-cb(x)f'(x-ct) = c[a'(x)f(x-ct) + a(x)f'(x-ct)] \tag{2.5}$$

We then substitute equation (2.5) into equation (2.4) to get

$$-ca(x)f'(x-ct) = cb'(x)f(x-ct) - c[a'(x)f(x-ct) + a(x)f'(x-ct)] - \mu(x)a(x)f(x-ct)$$
$$\text{or} \tag{2.6}$$
$$0 = cb'(x)f(x-ct) - ca'(x)f(x-ct) - \mu(x)a(x)f(x-ct) \tag{2.7}$$

Given our problem setting, we exclude $f(x-ct) = 0$. So we have

$$cb'(x) - ca'(x) - \mu(x)a(x) = 0. \tag{2.8}$$

Note that the specific choice of $b(x)$ does not matter as long as its first derivative is not 0. Because when we transform Equation (2.4) and Equation (2.5) back to the second-order wave equation (i.e. the second-order wave equation after substituting

$u(x,t) = a(x)f(x-ct)$ into (1.1)), all terms related to $b(x)$ or $b'(x)$ will be cancelled out. Thus, we can simply let $b(x) = -a(x)$ since we also do not want $\mu(x)a(x) = 0$.

Finally, equation (2.8) is just a simple ordinary differential equation. Assume $\mu(x)$ is integrable for $1 < x < 1+\delta$, we have the solution

$$a(x) = Ae^{-\frac{1}{2c}\int \mu(x)\mathrm{d}x}, \tag{2.9}$$

for some constant A. Since $\mu(x)$ is positive for $1 < x < 1+\delta$, our solution of (1.1) has an exponential decay in amplitude.

2.3 Damping Models

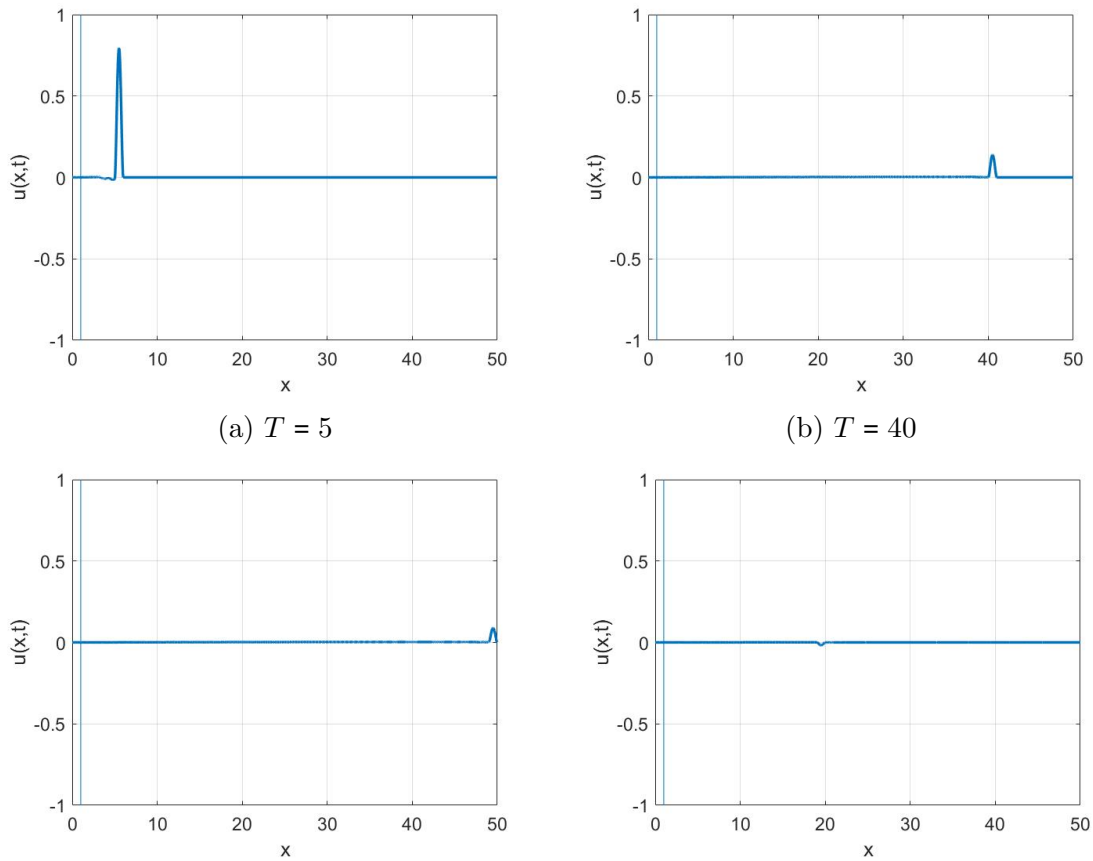
2.3.1 Piece-wise Constant Damping Coefficient

An intuitive choice of the damping coefficient $\mu(x)$ is the piece-wise constant function such as

$$\mu(x) = \begin{cases} 0 & 0 \le x \le 1 \\ \mu_0 & 1 < x \le 1+\delta \end{cases} \tag{2.10}$$

for some positive constant μ_0 and δ. However, imposing a sudden nonzero damping coefficient in the wave equation is like putting a boundary condition next to the domain of interest, so we need to be careful with the reflected wave caused by the change of damping coefficient. A relatively wise choice is to use a very small value of μ_0 and make δ (i.e. the thickness of the buffer area) very large. In other words, in order to minimize the reflected wave caused by the change of the damping coefficient, we want t be the main factor that controls the decay of the amplitude rather than μ_0.

The numerical example in Section 4.2 shows that we have an optimal choice of $\mu_0 = 0.1$ for a large $\delta = 49$. Figure 2.1 demonstrates the results under the optimized damping term in Section 4.2.



(a) $T = 5$ (b) $T = 40$

(c) $T = 49$ (d) $T = 80$

Figure 2.1: Movement of the wave governed by (1.2) with a piece-wise constant damping coefficient (2.10) and $\mu_0 = 0.1$ under the initial conditions in Section 4.2. Here $c = 1$.

A constant damping coefficient in $1 < x \leq 1+\delta$ allow us another approach to verify the decay rate in Section 2.2. Using separation of variables to solve for the solution in $1 < x < 1 + \delta$, i.e. the solution of equation $u_{tt} - c^2 u_{xx} + \mu_0 u_t = 0$, we have

$$T''(t) + \mu_0 \cdot T'(t) - \lambda \cdot c^2 \cdot T(t) = 0, \tag{2.11}$$

where $u(x,t) = T(t)X(x)$ and $\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} + \frac{\mu_0}{c^2}\frac{T'(t)}{T(t)} = \lambda$. The solution of Equation (2.11) is

$$T(t) = e^{-\frac{\mu_0}{2}t} \cdot \left(C_1 \cdot \sin\left(\frac{t}{2}\sqrt{\mu_0^2 + 4c^2\lambda}\right) + C_2 \cdot \cos\left(\frac{t}{2}\sqrt{\mu_0^2 + 4c^2\lambda}\right)\right), \tag{2.12}$$

for some constants C_1 and C_2.

Regardless of the constants, the term $e^{-\frac{\mu_0}{2}t}$ in Equation (2.12) implies a exponential decay of the amplitude when the wave travels in the buffer area. This is consistent with our conclusion in Section 2.2 because the space coordinate of the peak of the wave can be written as $ct + D$ for some constant D. So, Equation (2.9) becomes

$$a(x) = Ae^{-\frac{1}{2c}\int \mu(x)\mathrm{d}x} = Ae^{-\frac{1}{2c}\cdot\mu_0 x} = Be^{-\frac{\mu_0}{2}t}, \tag{2.13}$$

for some constant B.

Figure 2.2 shows that in our numerical experiment in Section 4.2, the result has decay rate almost identical to $e^{-\frac{\mu_0}{2}t}$.

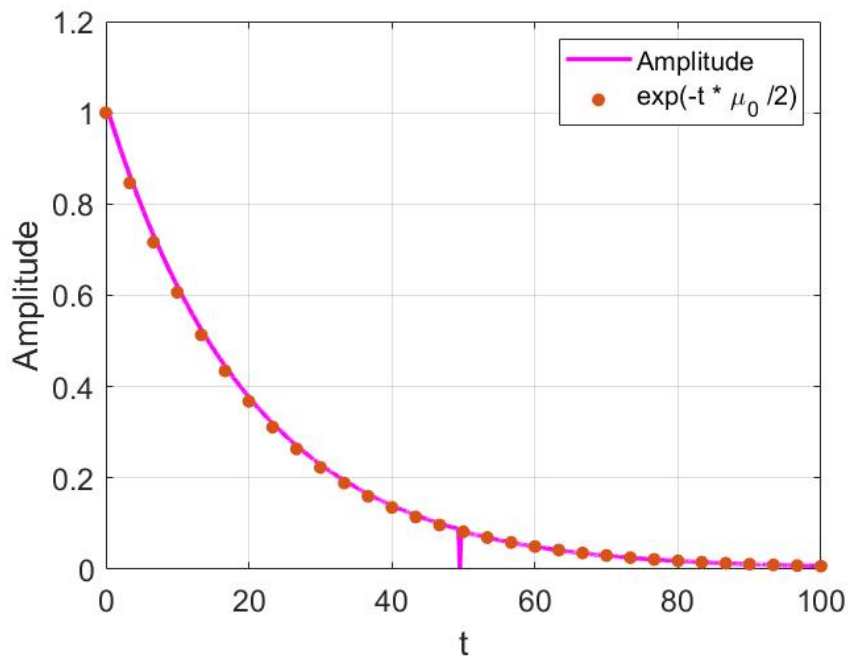


Figure 2.2: Amplitude of the wave governed by Equation (1.2) with the piece-wise constant damping coefficient $\mu_0 = 0.1$ under the initial conditions in Section 4.2. The sudden drop at $t = 50$ is caused by the boundary condition. Here $c = 1$.

2.3.2 Smooth Damping Coefficient

Continuing the idea from the last section (Section 2.3.1) that a sudden change in the damping coefficient acts like a boundary condition in the solution, we may use a gradually increasing function to achieve a better result, such as Figure 2.3.

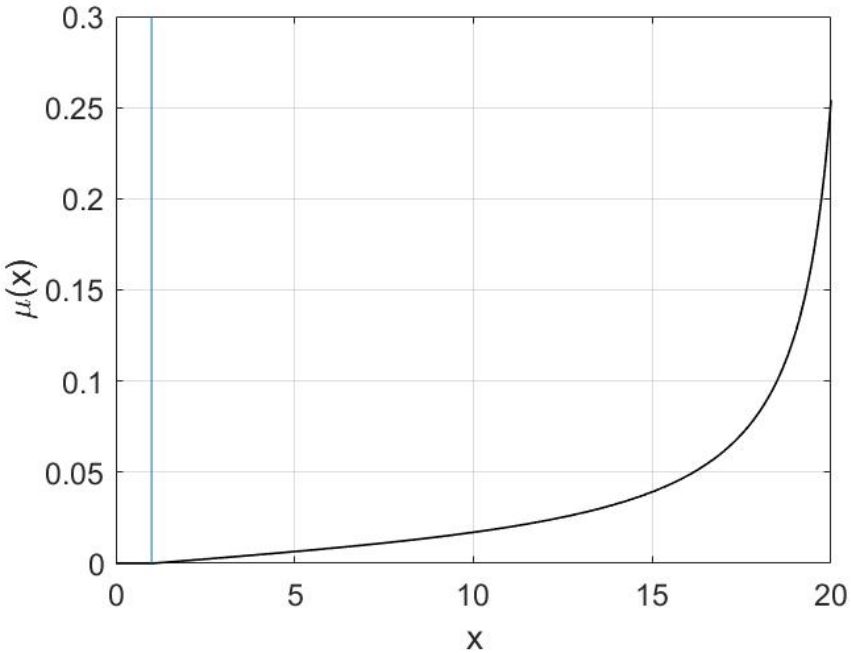
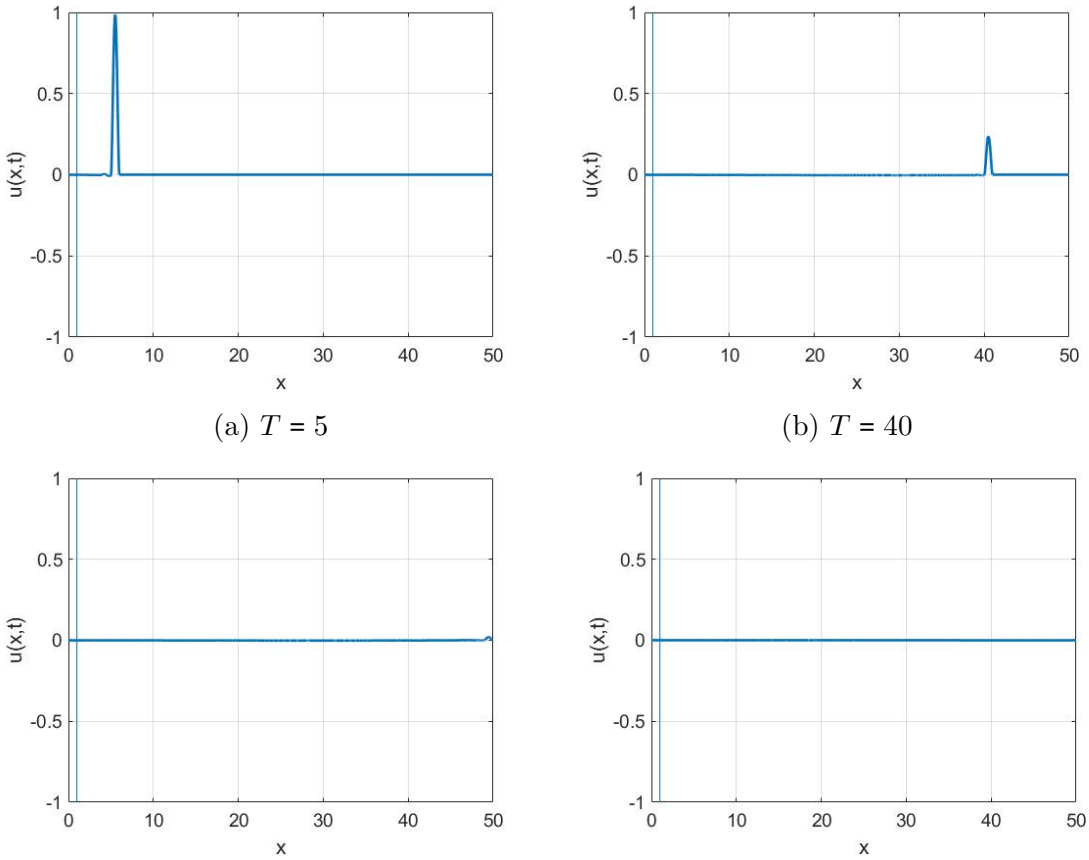


Figure 2.3: An example of an ideal smooth damping coefficient. Here μ is zero in $0 \leq x \leq 1$ and slowly increases in magnitude at the front portion of the buffer area, but quickly rise at the late part of the buffer area.

One function that can achieve our purpose is the tangent function, i.e.

$$\mu(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ \mu_0 \cdot \tan\left(\frac{\pi}{2} \cdot \frac{x-1}{1+\delta}\right) & 1 < x \leq 1 + \delta \end{cases}, \tag{2.14}$$

where μ_0 is also a constant, same as the one in Section 2.3.1, but the scale can be much smaller. Figure 2.4 demonstrates the results under the optimized damping term in Section 4.2. A more detailed comparison between the two choices of the damping coefficient is illustrated in Section 4.2. However, by only looking at Figure 2.1 and Figure 2.4, the benefits of using a smooth damping coefficient is already very obvious.



(a) $T = 5$ (b) $T = 40$

(c) $T = 49$ (d) $T = 80$

Figure 2.4: Movement of the wave governed by (1.3) with a smooth damping coefficient (2.14) and $\mu_0 = 0.08$ under the initial conditions in Section 4.2.

2.4 Transformation of the Damped Equation

As explained in the Section 2.3.1, the amplitude of any reflected waves from the right end of the domain at $x = 1 + \delta$ can be made negligible by choosing an appropriate value for $\mu(x)$ and a suitably large value for δ; in the examples above we used $\delta = 50$. However, such a large value for δ is a bit inconvenient. Can we obtain similar results with a much smaller value for δ? In this section we examine a simple change of spatial variable in equation (1.2) that makes the buffer region much smaller, at the expense of making the damping coefficient $\mu(x)$ more singular.

Let $\phi(x) : [0, 1 + \delta] \to [0, 1 + \epsilon]$ be the function for the change of variable, where δ, ϵ are constants we pick and $0 < \epsilon << \delta$. We want ϕ to map the interval $0 < x < 1 + \delta$ to $0 < x < 1 + \epsilon$ in a one-to-one and C^2 fashion, while fixing the interval $(0, 1)$. Thus ϕ should satisfy

$$\begin{cases} \phi(x) & = x \quad \text{for } 0 \le x \le 1 \\ \phi(1+\delta) & = 1 + \epsilon \\ \phi'(1) & = \frac{\mathrm{d}}{\mathrm{d}x} x = 1 \\ \phi''(1) & = \frac{\mathrm{d}^2}{\mathrm{d}x^2} x = 0 \end{cases} \tag{2.15}$$

with $\phi'(x) > 0$.

One choice is

$$\phi(x) = \begin{cases} x, & 0 \le x < 1 \\ 1 + a \cdot \arctan\left(\frac{x-1}{a}\right), & 1 \le x \le 1 + \delta \end{cases}, \tag{2.16}$$

where a is the solution of

$$\begin{aligned} 1 + a \cdot \arctan\left(\frac{\delta}{a}\right) &= 1 + \epsilon. \\ a \cdot \arctan\left(\frac{\delta}{a}\right) &= \epsilon. \end{aligned} \tag{2.17}$$

Equation (2.17) has a unique solution in a for any $0 < \epsilon < \delta$. To see this, note that because $a \cdot \arctan\left(\frac{\delta}{a}\right)$ is bounded between 0 and δ for $0 < a < \infty$, and

$$\begin{aligned} \lim_{a \to 0^+} a \cdot \arctan\left(\frac{\delta}{a}\right) &= 0 \\ \lim_{a \to \infty} a \cdot \arctan\left(\frac{\delta}{a}\right) &= \delta. \end{aligned}$$

Let $\psi(a) = a \cdot \arctan\left(\frac{\delta}{a}\right)$ and $\eta(b) = \arctan(b) - \frac{b}{b^2+1}$. Then,

$$\psi'(a) = \eta\left(\frac{\delta}{a}\right). \tag{2.18}$$

Because $\eta(0) = 0$ and

$$\eta'(b) = \frac{2b^2}{(b^2+1)^2} > 0 \text{ for } b > 0,$$

$\eta(b) > 0$ for all $b > 0$. Since $\frac{\delta}{a} > 0$, ψ' is always positive and so $\psi(a)$ is monotonically increasing with $\psi(a) > 0$ for all $a \in (0, \infty)$. In this case, for a given δ and ϵ, the solution of equation (2.17) (which is simply $\psi(a) = \epsilon$) exists and is unique.

We can quickly check if our $\phi(x)$ in (2.16) satisfies conditions in (2.15) by substituting $x = 1$ and $x = 1 + \delta$ into $\phi(x)$,$\phi_x(1)$, and $\phi_{xx}(1)$:

$$\phi(1) = 1 + a \cdot \arctan\left(\frac{1-1}{a}\right) = 1 + a \cdot 0 = 1 \tag{2.19}$$

$$\phi_x(1) = \left.\frac{a^2}{(x-1)^2 + a^2}\right|_{x=1} = \frac{a^2}{0 + a^2} = 1 \tag{2.20}$$

$$\phi_{xx}(1) = \left.\frac{2a^2(1-x)}{((x-1)^2 + a^2)^2}\right|_{x=1} = \frac{2a^2 \cdot 0}{(0 + a^2)^2} = 0, \tag{2.21}$$

and $\phi(1 + \delta) = 1 + \epsilon$ is satisfied by the choice of a in (2.17).

We now use ϕ to perform a change of variable in equation (1.2). Let $y = \phi(x)$, so

$$x = \phi^{-1}(y) = \begin{cases} y, & 0 \le y < 1 \\ 1 + a \cdot \tan\left(\frac{y-1}{a}\right), & 1 \le y \le 1 + \epsilon \end{cases}, \tag{2.22}$$

where a is still the solution of (2.17).

We want to write equation (1.2) with damping coefficient (2.10) in terms of $v(y,t)$ where $v(y,t) = u(x,t)$. Since for $0 \le x \le 1$, equation (1.2) is just the undamped wave equation and ϕ fixes $(0,1)$, $v(y,t)$ satisfies the same equation there. So we just consider $1 < x \le 1 + \delta$. With $y = \phi(x)$, we have

$$u_x = \frac{\partial}{\partial x} u = \frac{\partial}{\partial y} v \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = v_y \cdot \phi(x)_x \tag{2.23}$$

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x} u_x = \frac{\partial}{\partial x}(v_y \cdot \phi(x)_x) \\ &= \left(\frac{\partial}{\partial y} v_y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\right) \cdot \phi(x)_x + v_y \cdot \left(\frac{\partial}{\partial x}\phi(x)_x\right) \\ &= v_{yy} \cdot (\phi_x)^2 + v_y \cdot \phi_{xx}. \end{aligned} \tag{2.24}$$

With (2.20), (2.21) and (2.22), we further have

$$
\begin{aligned}
u_{xx} &= v_{yy} \cdot \Big(\frac{a^2}{(x-1)^2 + a^2}\Big)^2 + v_y \cdot \frac{2a^2(1-x)}{((x-1)^2 + a^2)^2} \\
&= v_{yy} \cdot \frac{a^4}{((1 + a \cdot \tan(\frac{y-1}{a}) - 1)^2 + a^2)^2} + v_y \cdot \frac{2a^2(1 - (1 + a \cdot \tan(\frac{y-1}{a})))}{((1 + a \cdot \tan(\frac{y-1}{a}) - 1)^2 + a^2)^2} \\
&= v_{yy} \cdot \frac{a^4}{a^4(\tan^2(\frac{y-1}{a}) + 1)^2} - v_y \cdot \frac{2a^2(a \cdot \tan(\frac{y-1}{a}))}{a^4(\tan^2(\frac{y-1}{a}) + 1)^2} \\
&= v_{yy} \cdot \frac{1}{\sec^4(\frac{y-1}{a})} - v_y \cdot \frac{2}{a} \cdot \frac{\tan(\frac{y-1}{a})}{\sec^4(\frac{y-1}{a})} \\
&= \cos^2\left(\frac{y-1}{a}\right) \cdot \Big(v_{yy} \cdot \cos^2\left(\frac{y-1}{a}\right) - v_y \cdot \frac{2}{a} \cdot \cos^2\left(\frac{y-1}{a}\right) \cdot \sin\left(\frac{y-1}{a}\right)\Big) \\
&= \cos^2\left(\frac{y-1}{a}\right) \cdot \Big(\frac{\partial}{\partial y} v_y \cdot \cos^2\left(\frac{y-1}{a}\right) - v_y \cdot \frac{\partial}{\partial y} \cos^2\left(\frac{y-1}{a}\right)\Big) \\
&= \cos^2\left(\frac{y-1}{a}\right) \cdot \frac{\partial}{\partial y}\Big(v_y \cdot \cos^2\left(\frac{y-1}{a}\right)\Big).
\end{aligned} \tag{2.25}
$$

Because we do not transform the time coordinate, we maintain $u_t = v_t$ and $u_{tt} = v_{tt}$. The damping coefficient (2.10) becomes

$$
\sigma(y) = \mu(x) = \mu(\phi^{-1}(y)) = \begin{cases} 0 & 0 \le \phi^{-1}(y) \le 1 \\ \mu_0 & 1 < \phi^{-1}(y) \le 1 + \delta \end{cases} = \begin{cases} 0 & 0 \le y \le 1 \\ \mu_0 & 1 < y \le 1 + \epsilon \end{cases} \tag{2.26}
$$

Thus, we have the transformed PDE of (1.2) based on (2.25) and (2.26)

$$
v_{tt} - \left(c \cdot \cos\left(\frac{y-1}{a}\right)\right)^2 \cdot \frac{\partial}{\partial y}\left(v_y \cdot \cos^2\left(\frac{y-1}{a}\right)\right) + \sigma(y) v_t = 0 \tag{2.27}
$$

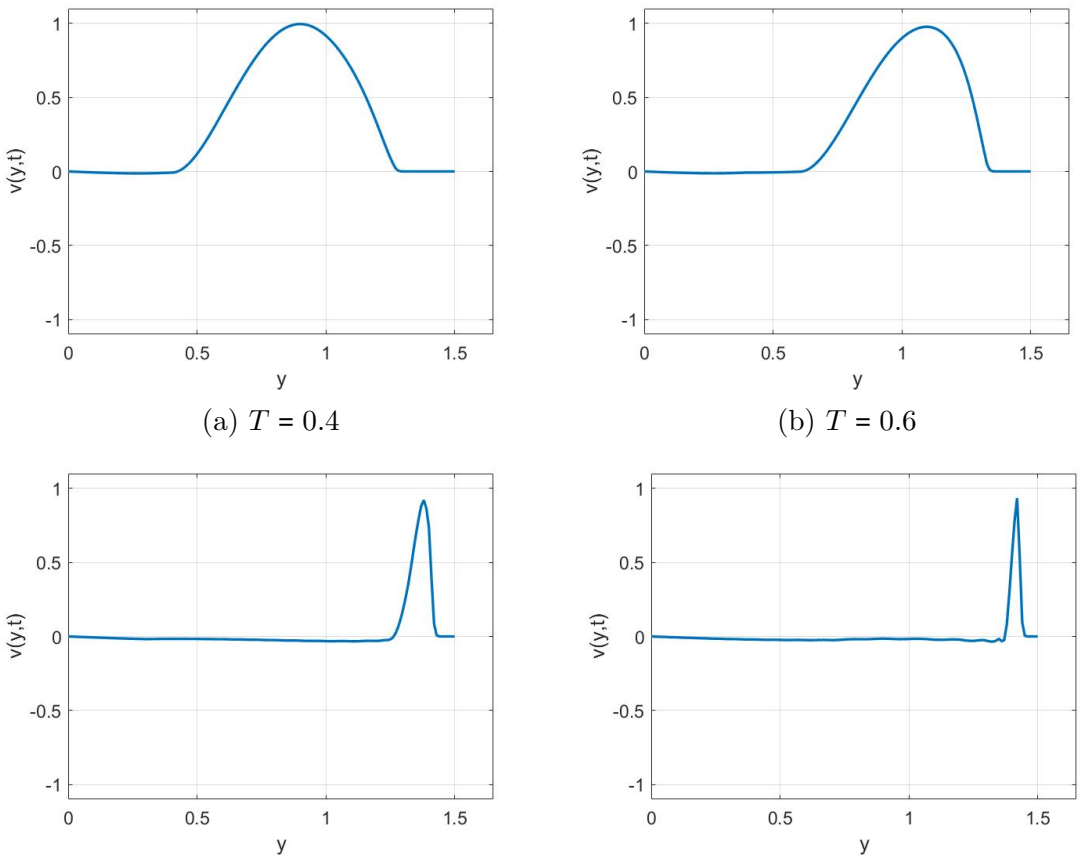
$$
\begin{cases} v(0,t) = 0 \\ v(1+\epsilon, t) = 0 \end{cases}, \tag{2.28}
$$

where

$$
\sigma(y) = \begin{cases} 0 & 0 \le y \le 1 \\ \mu_0 & 1 < y \le 1 + \epsilon \end{cases} \tag{2.29}
$$

In equation (2.27), we can see when the wave towards the boundary, $\cos\left(\frac{y-1}{a}\right)$ decreases as $\frac{y-1}{a}$ increases from 0 to $\frac{\pi}{2}$. This explains why our transformation can

maintain the same scale of errors with a very small buffer area: the significant decrease of speed wave in the buffer area means there is sufficient time for the exponential decay rate to dramatically lower the amplitude of the wave even when the buffer area is small. Figure 2.5 illustrates this process.



(a) $T = 0.4$ (b) $T = 0.6$

(c) $T = 1.3$ (d) $T = 1.7$

Figure 2.5: Movement of the wave governed by (2.27) with piece-wise constant damping coefficient (2.10) and $\mu_0 = 0.2$ under the initial conditions in Section 4.2.

2.4.1 Theoretically Perfect Cloaking

If we further look at the case when $\delta \to \infty$, then we can have a situation in which no reflection from the right side of the interval occurs, for any damping $\mu > 0$. Indeed, we can use an arbitrarily small constant μ in this case, to minimize reflections at $x = 1$. In conjunction with the transformation embodied by ϕ, this should allow us to design a buffer region $1 < x < 1 + \epsilon$ with suitable coefficients that gives arbitrarily small reflections. It may be possible to make a properly scaled limiting argument for fixed ϵ as $\delta \to \infty, \mu_0 \to 0$, that gives a perfectly reflectionless buffer zone, but we have not yet done this.

However, in such a perfect cloaking situation, $a = \frac{2\epsilon}{\pi}$ satisfy the Equation (2.17) (i.e. $\frac{\epsilon}{2} = \frac{\pi}{2}$) and equation (2.27) becomes

$$v_{tt} - \left(c \cdot \cos\left(\frac{y-1}{2\epsilon}\pi\right)\right)^2 \cdot \frac{\partial}{\partial y}\left(v_y \cdot \cos^2\left(\frac{y-1}{2\epsilon}\pi\right)\right) + \sigma(y)v_t = 0. \tag{2.30}$$

Since $y \to 1 + \epsilon$ implies $\cos\left(\frac{y-1}{2\epsilon}\pi\right) \to 0$, when the wave moves close to the outer boundary, the wave speed goes to 0. In other words, the wave stays in the buffer area forever until its amplitude decay to 0. If we have a material that, no matter how small the damping it provides, can lower the wave speed to 0 eventually, it can provide a perfect cloaking.

Chapter 3

Perfectly Matched Layers

Let us for now consider another method for cloaking: perfectly matched layers (PML). The idea between PML and our approach is very similar, that is, adding a layer of buffer or absorbing material outside the medium in which the wave travels so that the amplitude of the wave decays to zero, but the wave in a particular domain of interest is not affected. However, instead of adding a damping term directly into the wave equation, PML adds the damping term during the coordinate transformation. The correctness, well-posedness, and stability of PML are shown in [1]. In this chapter, we provide the PML formulation for our wave equation and a comparison between our method and PML.

3.1 Perfectly Matched Layer Formulation

We can transfer Equation (1.1) to a hyperbolic system

$$\begin{cases} u_t &= c \cdot w_x \\ w_t &= c \cdot u_x \end{cases}, \tag{3.1}$$

where $w(x,t)$ is an auxiliary function. The existence of the $w(x,t)$ is proved by [6] and [8].

Note that hyperbolic system (3.1) can be also written as

$$\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial}{\partial t}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ w \end{pmatrix}, \tag{3.2}$$

where $\mathbf{U} = (u, w)^\top$. As stated in [7], by assuming the isotropic medium, $\mathbf{U}(x,t)$ is the superposition of planewaves (in our case, moving to the right), we have

$$\mathbf{U}(x,t) = \sum_{k=1}^{\infty} \mathbf{A}_k e^{i(kx-ct)}, \tag{3.3}$$

where $\mathbf{A}_k$ are some constants. By using linearity we can focus on a single wave $\mathbf{A}e^{i(kx-ct)}$ and see that we have

$$\frac{\partial \mathbf{U}}{\partial t} = -ick\mathbf{U}. \tag{3.4}$$

That is to say, for these fixed frequency waves we have

$$\begin{cases} u_t &= cw_x = -icku \\ w_t &= cu_x = -ickw. \end{cases} \tag{3.5}$$

[9, 7] shows that, in practice, PML is a simple substitution in space coordinate(s):

$$\frac{\partial}{\partial x} \to \frac{1}{1 + i\frac{\mu(x)}{kc}} \frac{\partial}{\partial x}, \tag{3.6}$$

where $\mu(x)$ is another auxiliary function that works the same as the one in Section 2.1. We use $\frac{\mu(x)}{kc}$ instead of $\mu(x)$ only for computational convenience purpose.

Thus, applying Equation (3.6) into Equation (3.5), we have

$$\begin{aligned} \begin{cases} \frac{c}{1+i\frac{\mu(x)}{kc}} w_x &= -ikcu \\ \frac{c}{1+i\frac{\mu(x)}{kc}} u_x &= -ikcw \end{cases} &\Rightarrow \begin{cases} cw_x &= (-ikcu)\left(1 + i\frac{\mu(x)}{kc}\right) \\ cu_x &= (-ikcw)\left(1 + i\frac{\mu(x)}{kc}\right) \end{cases} \\ &\Rightarrow \begin{cases} cw_x &= -ikcu + \mu(x)u \\ cu_x &= -ikcw + \mu(x)w \end{cases} \end{aligned} \tag{3.7}$$

We then substitute Equation (3.5) into Equation (3.7) to get

$$\begin{cases} cw_x &= u_t + \mu(x)u \\ cu_x &= w_t + \mu(x)w \end{cases} \tag{3.8}$$

or

$$\begin{cases} u_t &= cw_x - \mu(x)u \\ w_t &= cu_x - \mu(x)w \end{cases} \tag{3.9}$$

We do not have any complex terms in (3.9). Compare to the system of the first-order wave equation in our method (2.2), we can see the only difference is the damping term in the second equation. There is no problem in applying the two damping models in Section 2.3 for the $\mu(x)$ in (3.9).

As with our damping approach, we will take $\mu(x) \equiv 0$ for $0 < x < 1$, and $\mu(x) > 0$ for $x > 1$.

The function $u(x,t)$ has the same initial conditions and boundary conditions in (1.3). Because we only interested in the initial wave travels to the right, for $0 < x < 1$, we have $u(x,t) = f(x-ct)$. Thus, $u_t(x,t) = -cf'(x-ct)$ and it can be substituted into the first equation of (3.9) to get

$$\begin{aligned} -cf'(x) &= cw_x(x,0) \\ w_x(x,0) &= -f'(x) \\ w(x,0) &= -f(x). \end{aligned} \tag{3.10}$$

The boundary conditions of $u(x,t)$, $u(0,t) = u(1+\delta,t) = 0$, can be substituted into the first equation of (3.9) to get the boundary conditions of $w(x,t)$: $w_x(0,t) = w_x(1+\delta,t) = 0$.

Thus, the PML formulation in our problem is

$$\begin{cases} u_t &= cw_x - \mu(x)u \\ w_t &= cu_x - \mu(x)w \end{cases} \tag{3.11}$$

$$\begin{cases} u(x,0) &= f(x) \\ w(x,0) &= -f(x) \\ u(0,t) &= u(1+\delta,t) = 0 \\ w_x(0,t) &= w_x(1+\delta,t) = 0 \end{cases}. \tag{3.12}$$

We should note that PML is, in principle, reflectionless at the $x = 1$ interface, due to the nature of the change of variables, unlike our damped approach. Waves moving to the right in the PML formulation are quickly attenuates, so truncating the region at $x = 1+\delta$ for large enough δ should prevent reflections. And, as we did with our damped approach above, one could perform a change of variable to make a smaller buffer region.

Chapter 4

Numerical Experiments

4.1 Finite Difference Scheme

We use a finite difference scheme to solve the relevant PDEs numerically. On a spatial interval $0 < x < b$ we take stepsize $\Delta x = b/N$ and set $x_i = i\Delta x$ for $0 \le i \le N$. For the time interval $0 < t < T$ we take $\Delta t = T/M$ with $t_j = j\Delta j$, $0 \le j \le M$. For $u(x,t)$ in i-th space and $j-th$ time grid, the finite difference method for equation (1.1) is

$$u_{tt}(x_i,t_j) \approx \frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{\Delta t^2} \tag{4.1}$$

$$u_{xx}(x_i,t_j) \approx \frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{\Delta x^2} \tag{4.2}$$

$$u_x(x_i,t_j) \approx \frac{u(x_{i+1},t_j) - u(x_{i-1},t_j)}{2\Delta x} \tag{4.3}$$

Applying von Neumann stability analysis, we have $|\lambda| \le \frac{\Delta t}{\Delta t + \mu(x)}$ under $c\frac{\Delta t}{\Delta x} \le 1$, where λ is the growth rate. So our scheme is stable as long as we have non-negative $\mu(x)$, which exactly fits the definition of $\mu(x)$.

Similarly, we use central difference method for $u_x(x,t)$ and $w_x(x,t)$ in the PML formulation (3.9) but we use backward difference method for $u_t(x,t)$ and $w_t(x,t)$, since both forward and central difference method on time derivative will both be unconditionally unstable due the introduction of damping term in the von Neumann stability analysis.

4.2 Numerical Example

We now use an example to demonstrate the two models of damping coefficients in Section 2.3 in our approach and the PML scheme. All PDE's in this section are solved with time step 0.01, space step 0.02, and time $T = 100$. We set initial wave

$$f(x) = \begin{cases} 16\left(x - \frac{1}{2}\right)^4 - 8\left(x - \frac{1}{2}\right)^2 + 1 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases} \tag{4.4}$$

for both (1.3) and (3.12) and initial velocity

$$g(x) = -f'(x) = \begin{cases} -64\left(x - \frac{1}{2}\right)^3 + 16x - 8 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases} \tag{4.5}$$

for (1.3).

We need a metric to measure the performance of each approach, that is, how much the solutions in the region $0 < x < 1$ in the damped or PML formulations differ from a perfectly reflectionless solution $u_0(x,t)$. The discrepancy will consists of reflections from the interface at $x = 1$ (which should occur in a relatively short time after $t = 0$) and longer term reflected waves from the far right boundary at $x = 1 + \delta$. With wave speed $c = 1$ and domain $0 < x < 50$, the short term reflections will occurs immediately, while the first reflection from the far right boundary will occur at about $t = 99$. We will thus measure the performance by computing the integral

$$I = \int_0^T \int_0^1 \left(u(x,t) - u_0(x,t)\right)^2 \mathrm{d}x\mathrm{d}t. \tag{4.6}$$

where $u_0(x,t)$ reflectionless solution $u_0(x,t) = f(x - ct)$ and $u(x,t)$ is the solution of two schemes with different damping models, and T is set to an appropriate value ($T \approx 2$ for to measure the immediate reflection, $T = 100$ to include the longer term reflection from the right).

By setting $\delta = 49$, Figure 4.1 illustrate that there exist optimal choices of the constant μ_0 in our approach, but the performance of PML seems independent to the choice of damping functions and μ_0, as expected. Table 4.1 shows that both damping models in our approach have smaller I compare to PML formulation, and smooth damping coefficient indeed has a much lower I than any others. However, if we look at Figure 4.2, we can see in PML formulation, not only there is no small reflected wave caused by the change of damping coefficient, but also the error in Figure 4.1 and Table 4.1 are from numerical computation error instead of error originated in the cloaking scheme.

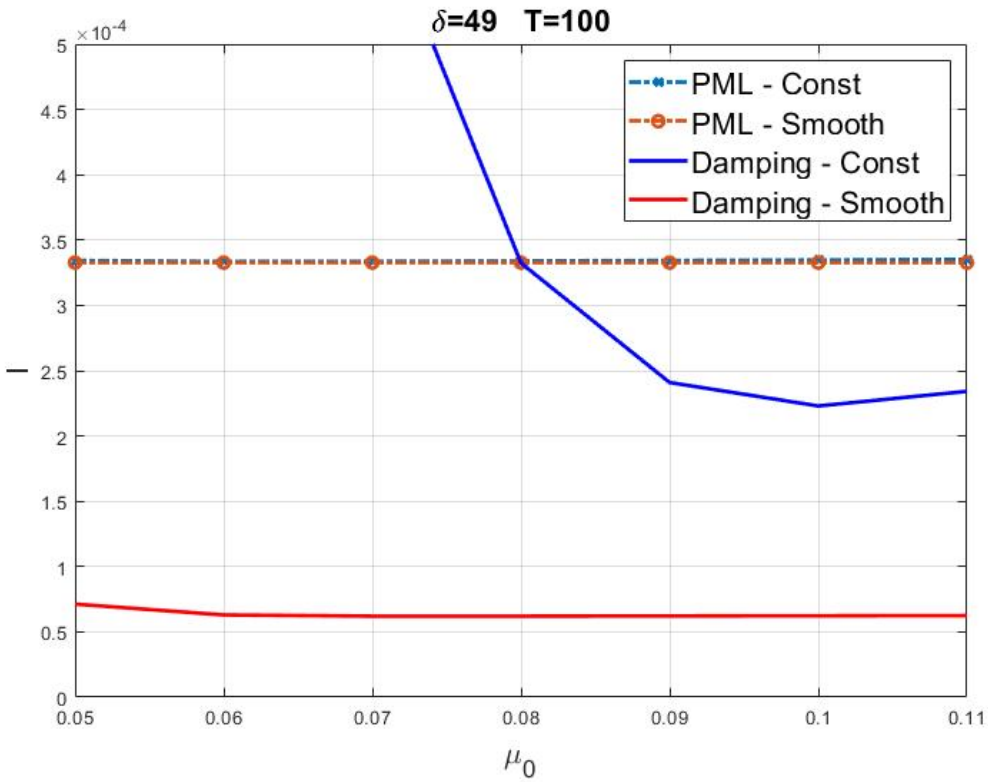
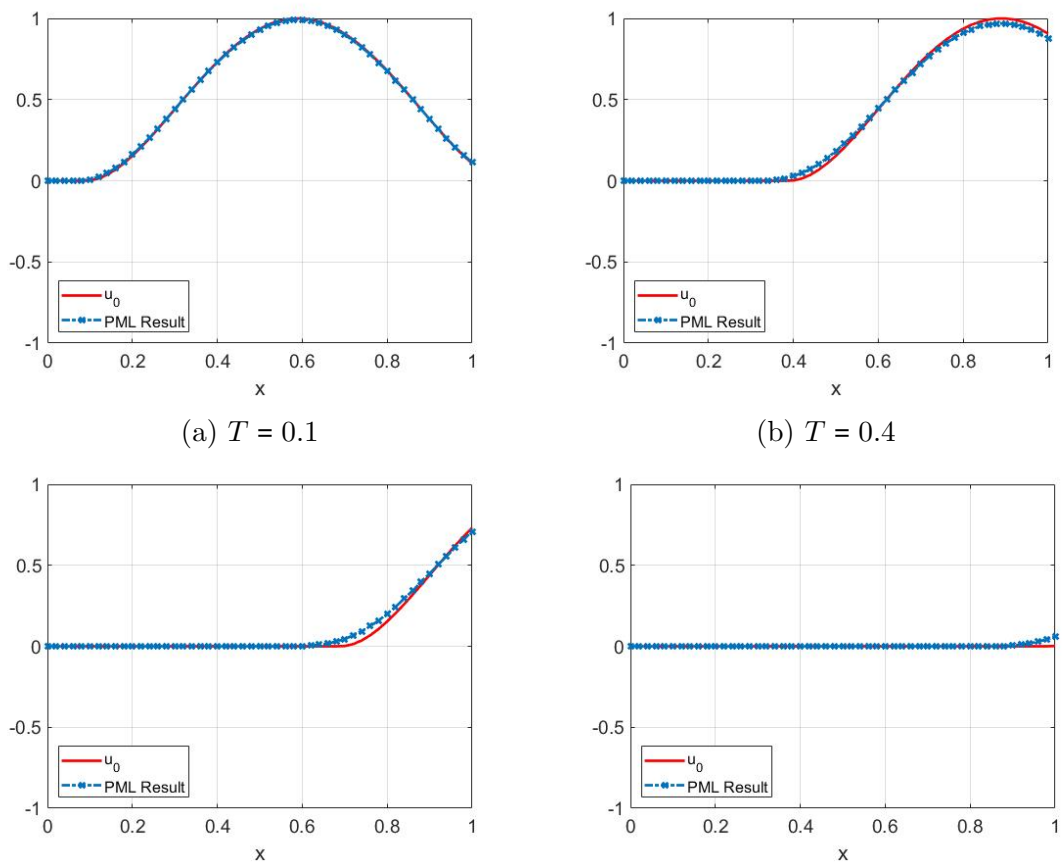


Figure 4.1: I Vs. μ_0 at δ = 49 and T = 100. "PML" stands for perfectly matched layer and "Damping" corresponds to our method. "Const" means the damping coefficient is a piece-wise constant function and "Smooth" indicates the damping coefficient is a smooth function from Section 2.3.2.

Table 4.1: Optimal μ_0's and I's

Method	Damping Coefficient	Optimal μ_0	Minimal I
Damping	Piece-wise Constant	0.10	$2.2297 \cdot 10^{-4}$
Damping	Smooth(Tangent)	0.08	$0.6206 \cdot 10^{-4}$
PML	Piece-wise Constant	N/A	$3.3357 \cdot 10^{-4}$
PML	Smooth(Tangent)	N/A	$3.3274 \cdot 10^{-4}$



(a) $T = 0.1$ (b) $T = 0.4$

(c) $T = 0.7$ (d) $T = 1.0$

Figure 4.2: Comparison between reflectionless situation (red wave $u_0(x,t)$), and PML solution with piece-wise constant damping coefficient, $\mu_0 = 0.08$.

Chapter 5

Discussion and Future Works

5.1 Discussion

Perfectly matched layers (PML) is a well-studied method in physics and widely applied to varied topics. The advantage of PML is that the wave can be perfectly absorbed analytically [1] and PML is relatively easy to implement. However, in this paper, we can see a proper damping term is probably an even more straightforward method to achieve the same goal as PML. It is also possible that the damping approach may, with a suitable scaling and limiting argument, lead to the PML form, at least in one space dimension. From the numerical perspective, our experiment in Chapter 4 shows that, with some reasonable numerical schemes, our approach can give an even better result than what PML can achieve.

However, did not find a way to achieve the perfect cloaking in our approach. Although our numerical experiments look promising, PML is still a better scheme for theoretical analysis on cloaking.

5.2 Future Works

In this thesis, we only discuss the one-dimensional wave equation. An extension to three-dimensional space would be interesting. Also, the existence of the optimal choice of the constant μ_0 in the damping coefficient may be proved analytically in the future. The proof of the decay rate in Section 2.2 is done by assuming the form of the solution; a more general proof will be useful.

On the other hand, the transformed equation in Section 2.4 can have some interesting physical interpretations related to relativity since a changing wave speed can be well interpreted under a curved space, which could be helpful on finding material to achieve the perfect cloaking in reality.

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