Cable Trench Problem

Matthew V Galati
Ted K Ralphs
Joseph C Hartman

magh@lehigh.edu

Department of Industrial and Systems Engineering
Lehigh University, Bethlehem, PA
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The Cable Trench Problem (CTP) is that of minimizing the cost of digging trenches and laying cable for a communications network given a central hub.

- Let \( G = (N, A) \) be a connected digraph with specified depot \( 0 \in N \).
- Define \( c_{ij} \) as the cost/weight (typically distance) on arc \((i, j)\).
- Define fixed charge variables (trench) \( x_{ij} \) as to whether or not to dig a trench between nodes \( i \) and \( j \).
- Define flow variables (cable) \( y_{ij} \) as to the amount of cable to lay between nodes \( i \) and \( j \).
A *node routing* is a directed subgraph $G'$ of $G$ satisfying
the following properties:

- $G'$ is (weakly) connected.
- The in-degree of each non-depot node is 1.
- It is a spanning arborescence.
- There is a unique path from the depot to each demand point (vertex).

**Cost Measures** (*least cost routing*)

- Sum the lengths of all arcs in $G'$.
- Sum the length of all paths from the depot.
- Some linear combination of these two.
IP Formulation

\[ \text{Min} \sum_{(i,j) \in A} \tau c_{ij}(x_{ij} + x_{ji}) + \gamma c_{ij}(y_{ij} + y_{ji}) \]

\[ \text{s.t.} \quad x(\delta(N \setminus \{i\})) = 1 \quad \forall i \in N \setminus \{0\} \quad (1) \]
\[ y(\delta(N \setminus \{i\})) - y(\delta(\{i\})) = d_i \quad \forall i \in N \setminus \{0\} \quad (2) \]
\[ y_{ij} \leq Mx_{ij} \quad \forall (i,j) \in A \quad (3) \]
\[ y_{ij} \geq 0 \quad \forall (i,j) \in A \quad (4) \]
\[ x_{ij} \in \{0, 1\} \quad \forall (i,j) \in A \quad (5) \]

where:

- (1) degree constraint
- (2) flow balance / demand constraint
- (3) capacity constraint
Complexity

This node routing problem is \textit{NP-complete} in general.

\textbf{Cable Trench Problem (}τ, γ > 0\textbf{)}

\begin{itemize}
  \item γ = 0 ⇒ Minimum Spanning Tree Problem.
  \item τ = 0 ⇒ Shortest Paths Tree Problem.
\end{itemize}

\textbf{Other special cases.}

\begin{itemize}
  \item γ = 0 and x(δ(\{i\})) = 1 ⇒ Traveling Salesman Problem (TSP).
  \item γ > 0 and x(δ(\{i\})) = 1 ⇒ Variable Cost TSP (VCTSP).
  \item x(δ(V \ \{0\})) = x(δ(\{0\})) = k ⇒ k-TSP.
\end{itemize}
Sample Spanning Trees

$\gamma = 0$

$\tau/\gamma = 10$

$\tau/\gamma = 0.1$

$\tau/\gamma = 0.001$
Theorem 1  Among all minimum spanning trees finding the one that minimizes the path length between a particular set of vertices $s$ and $t$ is NP-Complete.

Corollary 1  Among all minimum spanning trees finding the one that minimizes the total path length between a particular vertex $s$ and all other vertices in $V$ is NP-Complete.

Corollary 2  The Cable Trench Problem is NP-Complete.

Theorem 2  Among all shortest path trees rooted at $s$ finding the one that minimizes the total edge length is in $P$. 

Complexity
Previous Work

Vasko et. al - Kutztown University (to be published CAOR Nov 2001)

- Heuristic upper bound for all values of $\tau/\gamma$.

- The solution to CTP is a sequence of spanning trees such that as $\tau/\gamma$ increases, the total edge length strictly decreases each time another spanning tree becomes optimum and the total path length strictly increases.

- Total cost versus $\tau/\gamma$ is piecewise linear curve with strictly decreasing slopes.

Related Areas

- Fixed-Charge Network Flow
- Capacitated Network Design
Valid Inequalities

- Rounded Capacity Constraints

\[ \sum_{i \in S, j \in S} x_{ij} \geq \lceil d(S)/C \rceil \]

- Flow Linking Constraints

\[ y_{ij} \leq (C - d_i)x_{ij} \iff x_{ij} \geq \frac{y_{ij}}{C - d_i} \]

\[ y_{ij} - y(\delta\{j\}) \leq x_{ij}d_j \]

- Edge Cuts

\[ x_{ij} + x_{ji} \leq 1 \]
Flow linking constraints and edge cuts can be included explicitly or separated in polynomial time.

Separating rounded capacity constraints is NP-complete, but can be done effectively.
Implementation

The implementation uses SYMPHONY, a parallel framework for branch, cut, and price (relative of COIN/BCP).

In SYMPHONY, the user supplies:

- the initial LP relaxation, separation subroutines,
- feasibility checker, and other optional subroutines.

SYMPHONY handles everything else.

The source code and documentation are available from www.BranchAndCut.org

Workshop on COIN/BCP (TB42) - Laszlo Ladanyi, Ted Ralphs
Conclusions and Future Research

- The flow linking constraints help to force integrality.
- The edge cuts also help impose structure and integrality.

Future Research
- Generalizations of the model (different types of "trenches", different grades of "cable").
- Better (more specific) cuts for the case where $\tau/\gamma$ is not extreme.
- Take advantage of connections to other models.

Upper Bounds