Sequential Quadratic Optimization with Inexact Subproblem Solves

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involving joint work with

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Outline

Problem formulation

Our goal is to solve a constrained nonlinear optimization problem:

$$\begin{aligned} \min_x \ & f(x) \\ \text{s.t.} \ & c(x) = 0, \ \bar{c}(x) \leq 0. \end{aligned} \tag{NLP}$$

If (NLP) is infeasible, then at least we want to minimize constraint violation:

$$\min_x \ v(x), \ \text{where } v(x) := \|c(x)\|_1 + \|[\bar{c}(x)]^+\|_1. \tag{FP}$$

(A minimizer of (NLP) is always a minimizer of (FP).)

Sequential quadratic optimization

Advantages:

- "Parameter free" search direction computation (ideally)
- Strong global convergence properties and behavior
- Active-set identification $\Longrightarrow$ Newton-like local convergence

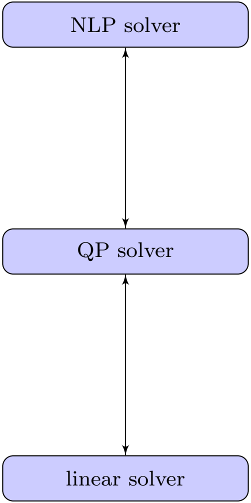
Disadvantages:

- No "best" way to handle inconsistent subproblems
- Quadratic subproblems (QPs) are expensive to solve exactly

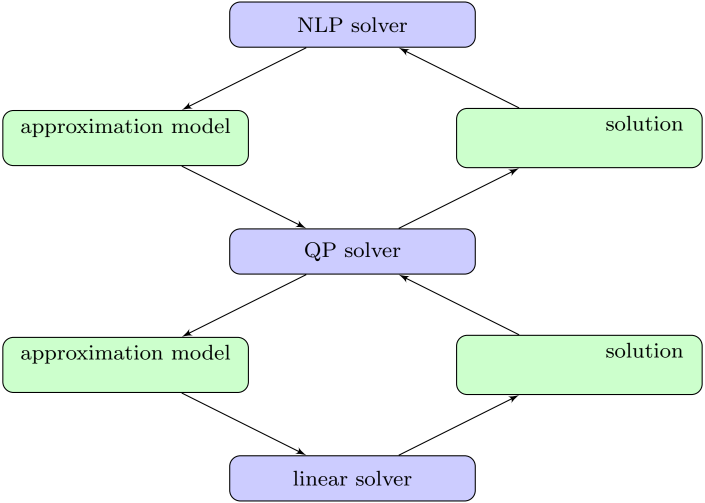
Open questions:

- Can we maintain the advantages of sequential quadratic optimization when the QP subproblems are solved inexactly?
- Can we maintain global and local convergence guarantees?

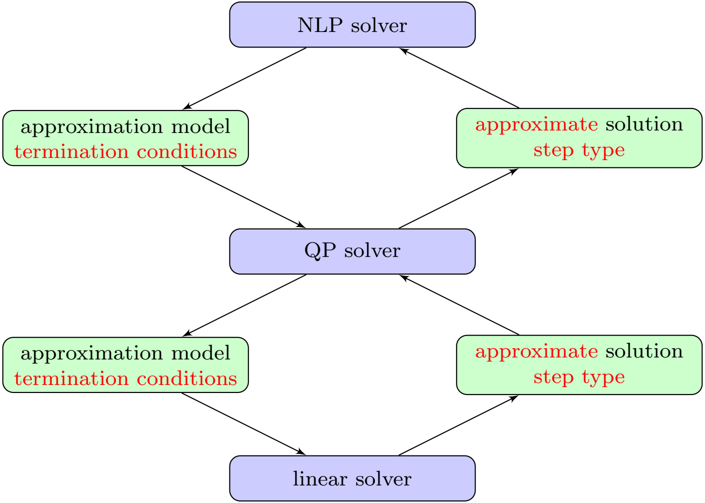
Algorithmic framework: Classic



Algorithmic framework: Detailed



Algorithmic framework: Inexact



Sequential quadratic optimization w/ inexactness

Contributions:

- Implementable termination conditions for inexact QP solves
- No specific QP solver required
- Global convergence guarantees (feasible and infeasible problems)
- Future work: Fast local convergence (feasible and infeasible problems)[1]

Algorithmic features:

- Allows "generic" inexactness in QP solutions
- Convex combination of "optimality" and "feasibility" steps
- Negative curvature handled with dynamic Hessian modifications
- Separate multipliers for (NLP) and (FP)
- Dynamic updates for penalty parameter and Lagrange multipliers

[1] Avoid using "Cauchy points" that only yield minimal progress for global convergence.

Fritz John and penalty functions

(NLP):

$$\min_x \; f(x)$$
$$\text{s.t. } c(x) = 0, \; \bar{c}(x) \leq 0$$

(FP):

$$\min_x \; v(x) := \left\| \begin{bmatrix} c(x) \\ [\bar{c}(x)]^+ \end{bmatrix} \right\|_1$$

Define the Fritz John (FJ) function

$$\mathcal{F}(x, y, \bar{y}, \mu) := \mu f(x) + c(x)^T y + \bar{c}(x)^T \bar{y}$$

and the ℓ_1-norm exact penalty function

$$\phi(x, \mu) := \mu f(x) + v(x).$$

$\mu \geq 0$ acts as objective multiplier/penalty parameter.

Optimality conditions

(NLP):

$$\min_x \ f(x)$$
$$\text{s.t. } c(x) = 0, \ \bar{c}(x) \leq 0$$

(FP):

$$\min_x \ v(x) := \left\| \begin{bmatrix} c(x) \\ [\bar{c}(x)]^+ \end{bmatrix} \right\|_1$$

(PP):

$$\min_x \ \phi(x, \mu) := \mu f(x) + v(x)$$

(FJ):

$$\mathcal{F}(x, y, \bar{y}, \mu) := \mu f(x) + c(x)^T y + \bar{c}(x)^T \bar{y}$$

KKT conditions for (FP) and (PP) expressed with residual

$$\rho(x, y, \bar{y}, \mu) := \begin{bmatrix} \mu g(x) + J(x)y + \bar{J}(x)\bar{y} \\ \min\{[c(x)]^+, e - y\} \\ \min\{[c(x)]^-, e + y\} \\ \min\{[\bar{c}(x)]^+, e - \bar{y}\} \\ \min\{[\bar{c}(x)]^-, \quad \bar{y}\} \end{bmatrix}$$

- FJ point:

$$\rho(x, y, \bar{y}, \mu) = 0, \ v(x) = 0, \ (y, \bar{y}, \mu) \neq 0$$

- KKT point:

$$\rho(x, y, \bar{y}, \mu) = 0, \ v(x) = 0, \ \mu > 0$$

- Infeasible stationary point:

$$\rho(x, y, \bar{y}, 0) = 0, \ v(x) > 0$$

Penalty function model and QP subproblem

(NLP):

$$\min_x \; f(x)$$
$$\text{s.t. } c(x) = 0, \;\; \bar{c}(x) \leq 0$$

(FP):

$$\min_x \; v(x) := \left\| \begin{bmatrix} c(x) \\ [\bar{c}(x)]^+ \end{bmatrix} \right\|_1$$

(PP):

$$\min_x \; \phi(x,\mu) := \mu f(x) + v(x)$$

(FJ):

$$\mathcal{F}(x,y,\bar{y},\mu) :=$$
$$\mu f(x) + c(x)^T y + \bar{c}(x)^T \bar{y}$$

KKT residual:

$$\rho(x,y,\bar{y},\mu)$$

Define a local model of $\phi(\cdot,\mu)$ at x_k:

$$l_k(d,\mu) := \mu(f_k + g_k^T d) + \|c_k + J_k^T d\|_1 + \|[\bar{c}_k + \bar{J}_k^T d]^+\|_1$$

Reduction in this model yielded by a given d:

$$\Delta l_k(d,\mu) := \Delta l(0,\mu) - \Delta l(d,\mu)$$

Subproblem of interest:

$$\min_d \; -\Delta l_k(d,\mu) + \tfrac{1}{2} d^T H d \qquad \text{(QP)}$$

$\Delta l_k(d,\mu) > 0$ implies d is a direction of strict descent for $\phi(\cdot,\mu)$ from x_k

Optimality conditions (for QP)

(NLP):

$$\min_x \; f(x) \quad \text{s.t. } c(x) = 0, \;\; \bar{c}(x) \le 0$$

(FP):

$$\min_x \; v(x) := \left\| \begin{bmatrix} c(x) \\ [\bar{c}(x)]^+ \end{bmatrix} \right\|_1$$

(PP):

$$\min_x \; \phi(x,\mu) := \mu f(x) + v(x)$$

(FJ):

$$\mathcal{F}(x,y,\bar{y},\mu) := \mu f(x) + c(x)^T y + \bar{c}(x)^T \bar{y}$$

KKT residual:

$$\rho(x,y,\bar{y},\mu)$$

Local model of ϕ at x_k:

$$l_k(d,\mu)$$

KKT conditions for (FP) and (PP) expressed with residual

$$\rho(x,y,\bar{y},\mu) := \begin{bmatrix} \mu g(x) + J(x)y + \bar{J}(x)\bar{y} \\ \min\{[c(x)]^+, e - y\} \\ \min\{[c(x)]^-, e + y\} \\ \min\{[\bar{c}(x)]^+, e - \bar{y}\} \\ \min\{[\bar{c}(x)]^-, \quad \bar{y}\} \end{bmatrix}$$

KKT conditions for (QP) expressed with

$$\rho_k(d,y,\bar{y},\mu,H) := \begin{bmatrix} \mu g_k + Hd + J_k y + \bar{J}_k \bar{y} \\ \min\{[c_k + J_k^T d]^+, e - y\} \\ \min\{[c_k + J_k^T d]^-, e + y\} \\ \min\{[\bar{c}_k + \bar{J}_k^T d]^+, e - \bar{y}\} \\ \min\{[\bar{c}_k + \bar{J}_k^T d]^-, \quad \bar{y}\} \end{bmatrix}$$

Exact solution of (QP):

$$\rho_k(d,y,\bar{y},\mu,H) = 0$$

Assumptions and well-posedness

Assumption

(1) *The functions f, c, and $\bar{c}$ and their first derivatives are bounded and Lipschitz continuous in an open convex set containing $\{x_k\}$ and $\{x_k + d_k\}$.*

(2) *The QP solver can solve* (QP) *arbitrarily accurately for any $\mu \geq 0$.*

Theorem (Well-posedness)

One of the following holds:

1. *iSQO terminates finitely with a KKT point or infeasible stationary point.*
2. *iSQO generates an infinite sequence of iterates*

$$\left(x_k, \begin{bmatrix} y_k' \\ \bar{y}_k' \end{bmatrix}, \begin{bmatrix} y_k'' \\ \bar{y}_k'' \end{bmatrix}, \mu_k\right) \text{ where } \begin{bmatrix} y_k' \\ y_k'' \end{bmatrix} \in [-e, e], \ \begin{bmatrix} \bar{y}_k' \\ \bar{y}_k'' \end{bmatrix} \in [0, e], \text{ and } \mu_k > 0.$$

Global convergence

Theorem (Global convergence)

One of the following holds:

(a) $\mu_k = \underline{\mu}$ *for some* $\underline{\mu} > 0$ *for all large* k *and either every limit point of* $\{x_k\}$ *corresponds to a KKT point or is an infeasible stationary point;*

(b) $\mu_k \to 0$ *and every limit point of* $\{x_k\}$ *is an infeasible stationary point;*

(c) $\mu_k \to 0$*, all limit points of* $\{x_k\}$ *are feasible, and, with*

$$K_\mu := \{k : \mu_{k+1} < \mu_k\},$$

every limit point of $\{x_k\}_{k \in K_\mu}$ *corresponds to an FJ point where the MFCQ fails.*

Corollary

If $\{x_k\}$ *is bounded and every limit point of this sequence is a feasible point at which the MFCQ holds, then* $\mu_k = \underline{\mu}$ *for some* $\underline{\mu} > 0$ *for all large* k *and every limit point of* $\{x_k\}$ *corresponds to a KKT point.*

"Direct" scenario

(NLP):

$$\min_x f(x) \quad \text{s.t. } c(x) = 0,\ \bar{c}(x) \le 0$$

(FP):

$$\min_x v(x) := \left\| \begin{bmatrix} c(x) \\ [\bar{c}(x)]^+ \end{bmatrix} \right\|_1$$

(PP):

$$\min_x \phi(x,\mu) := \mu f(x) + v(x)$$

(FJ):

$$\mathcal{F}(x,y,\bar{y},\mu) := \mu f(x) + c(x)^T y + \bar{c}(x)^T \bar{y}$$

KKT residuals:

$$\rho(x,y,\bar{y},\mu)$$
$$\rho_k(d,y,\bar{y},\mu,H)$$

Local model of ϕ at x_k:

$$l_k(d,\mu)$$

Terminate the QP solver when the solution $(d_k, y_{k+1}, \bar{y}_{k+1})$ of (QP) with $\mu = \mu_k$ satisfies

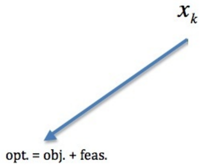
- $y_{k+1} \in [-e, e]$, $\bar{y}_{k+1} \in [0, e]$
- $\Delta l_k(d_k, \mu_k) \ge \theta \|d_k\|^2 > 0$ for $\theta \in (0,1)$
- $\|\rho_k(d_k, y_{k+1}, \bar{y}_{k+1}, \mu_k, H_k)\| \le \kappa \|\rho(x_k, y_k, \bar{y}_k, \mu_k)\|$

If

- $\Delta l_k(d_k, \mu_k) \ge \epsilon v_k$ for $\epsilon \in (0,1)$

then

- $d_k \leftarrow d_k$ is the search direction
- $\mu_{k+1} \leftarrow \mu_k$



"Reference" scenario

(NLP):

$$\min_x f(x)$$
$$\text{s.t. } c(x) = 0, \ \bar{c}(x) \leq 0$$

(FP):

$$\min_x v(x) := \left\| \begin{bmatrix} c(x) \\ [\bar{c}(x)]^+ \end{bmatrix} \right\|_1$$

(PP):

$$\min_x \phi(x, \mu) := \mu f(x) + v(x)$$

(FJ):

$$\mathcal{F}(x, y, \bar{y}, \mu) :=$$
$$\mu f(x) + c(x)^T y + \bar{c}(x)^T \bar{y}$$

KKT residuals:

$$\rho(x, y, \bar{y}, \mu)$$
$$\rho_k(d, y, \bar{y}, \mu, H)$$

Local model of ϕ at x_k:

$$l_k(d, \mu)$$

Terminate the QP solver when the solution $(d_k, y_{k+1}, \bar{y}_{k+1})$ of (QP) with $\mu = 0$ satisfies

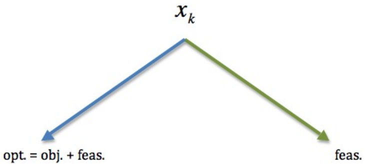
- $y_{k+1} \in [-e, e]$, $\bar{y}_{k+1} \in [0, e]$
- $\Delta l_k(d_k, 0) \geq \theta \|d_k\|^2$ for $\theta \in (0, 1)$
- $\|\rho_k(d_k, y_{k+1}, \bar{y}_{k+1}, 0, H_k)\| \leq \kappa \|\rho(x_k, y_k, \bar{y}_k, 0)\|$

If

- $\Delta l_k(d_k, \mu_k) \geq \epsilon \Delta l_k(d_k, 0)$ for $\epsilon \in (0, 1)$

then

- $d_k \leftarrow d_k$ is the search direction
- $\mu_{k+1} \leftarrow \mu_k$



“Combination” scenario

(NLP):

$$\min_x f(x)$$
$$\text{s.t. } c(x) = 0, \ \bar{c}(x) \leq 0$$

(FP):

$$\min_x v(x) := \left\| \begin{bmatrix} c(x) \\ [\bar{c}(x)]^+ \end{bmatrix} \right\|_1$$

(PP):

$$\min_x \phi(x, \mu) := \mu f(x) + v(x)$$

(FJ):

$$\mathcal{F}(x, y, \bar{y}, \mu) :=$$
$$\mu f(x) + c(x)^T y + \bar{c}(x)^T \bar{y}$$

KKT residuals:

$$\rho(x, y, \bar{y}, \mu)$$
$$\rho_k(d, y, \bar{y}, \mu, H)$$

Local model of ϕ at x_k:

$$l_k(d, \mu)$$

Choose $\tau \in [0, 1]$ as large as possible such that

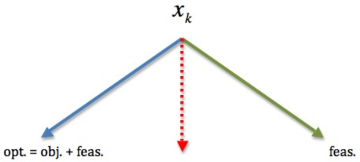
$$d_k \leftarrow \tau d_k + (1 - \tau) d_k$$

yields

$$\Delta l_k(d_k, 0) \geq \epsilon \Delta l_k(d_k, 0)$$

then choose $\mu_{k+1} < \mu_k$ such that

$$\Delta l_k(d_k, \mu_{k+1}) \geq \beta \Delta l_k(d_k, 0) \text{ for } \beta \in (0, 1)$$



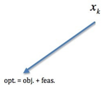
iSQO framework

repeat

(1) Check whether KKT point or infeasible stationary point has been obtained.

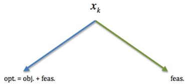
(2) Compute an inexact solution of (QP) with $\mu = \mu_k$.

(a) If "Direct" scenario occurs, then go to step 4.

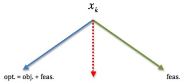


(3) Compute an inexact solution of (QP) with $\mu = 0$.

(a) If "Reference" scenario occurs, then go to step 4.



(b) If "Combination" scenario occurs, then go to step 4.



(4) Perform a backtracking line search to reduce $\phi(\cdot, \mu_{k+1})$.

endrepeat

Complicating factors

A few special cases make our actual algorithm slightly ;-) more complicated

- Landing on stationary points for $\phi(\cdot, \mu_k)$
  - We allow only a multiplier and/or penalty parameter update
- A tightened accuracy tolerance is needed in "combination" scenarios
  - We may require certain multipliers to be close to their bounds
  - (Think of identifying violated constraints)
- H_k and/or H_k may not be positive definite
  - We ask the QP solver to check the curvature along trial directions
  - (Dynamic inertia correction if trial curvature is too small/negative)

Actual algorithm involves six scenarios, but we have presented the "core" ideas

Implementation details

- Matlab implementation
- `BQPD` for QP solves with indefinite Hessians; see (Fletcher, 2000)
- *Simulated* inexactness by perturbing QP solutions
- Test set involves 307 `CUTEr` problems with
  - at least one free variable
  - at least one general (non-bound) constraint
  - at most 200 variables and constraints (because it's Matlab!)
- Termination conditions ($\epsilon_{tol} = 10^{-6}$ and $\epsilon_{\mu} = 10^{-8}$):

$$\begin{aligned}
\|\rho(x_k, y_k, \bar{y}_k, \mu_k)\|_\infty &\leq \epsilon_{tol} \quad \text{and} \quad v_k \leq \epsilon_{tol}; && \text{(Optimal)} \\
\|\rho(x_k, y_k, \bar{y}_k, 0)\|_\infty &= 0 \quad \text{and} \quad v_k > 0; && \text{(Infeasible)} \\
\|\rho(x_k, y_k, \bar{y}_k, 0)\|_\infty &\leq \epsilon_{tol} \quad \text{and} \quad v_k > \epsilon_{tol} \quad \text{and} \quad \mu_k \leq \epsilon_\mu && \text{(Infeasible)}
\end{aligned}$$

- Investigate performance of inexact algorithm with $\kappa = 0.01$, 0.1, and 0.5

Success statistics

Counts of termination messages for exact and three variants of inexact algorithm:

Termination message	Exact	Inexact		
		$\kappa = 0.01$	$\kappa = 0.1$	$\kappa = 0.5$
Optimal solution found	271	269	272	275
Infeasible stationary point found	4	3	2	2
Iteration limit reached	12	10	11	9
Subproblem solver failure	18	23	20	19

Termination statistics and reliability do not degrade with inexactness!

Inexactness levels

Observe "induced" relative residuals for QP solves:

$$\kappa_I := \frac{\|\rho_k\|}{\|\rho\|}$$

For problem j, we compute minimum ($\kappa_I(j)$) and mean ($\bar{\kappa}_I(j)$) values over run:

min	κ	$\kappa_{I,\text{mean}}$	$[0, 10^{-8})$	$[10^{-8}, 10^{-6})$	$[10^{-6}, 10^{-4})$	$[10^{-4}, 10^{-3})$	$[10^{-3}, 0.01)$	$[0.01, 0.1)$	$[0.1, 0.5)$	$[0.5, 1)$	$[1, \infty)$
$\kappa_I(j)$	0.01	3.5e-03	0	2	10	7	253	0	0	0	0
	0.1	2.8e-02	0	0	2	10	30	232	0	0	0
	0.5	8.8e-02	0	0	2	4	23	69	179	0	0
mean	κ	$\bar{\kappa}_{I,\text{mean}}$									
$\bar{\kappa}_I(j)$	0.01	7.3e-03	0	0	0	0	254	18	0	0	0
	0.1	6.9e-02	0	0	0	0	0	261	13	0	0
	0.5	3.5e-01	0	0	0	0	0	1	264	12	0

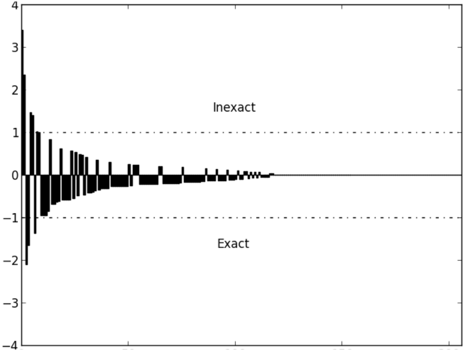
Relative residuals generally need only be moderately smaller than parameter κ!

Iteration comparison

Considering the logarithmic outperforming factor

$$r^j := -\log_2(\text{iter}^j_{\text{inexact}}/\text{iter}^j_{\text{exact}}),$$

we compare iteration counts of our inexact ($\kappa = 0.01$) and exact algorithms:



Iteration counts do not degrade significantly with inexactness!

Summary

Contributions:

- Developed, analyzed, and experimented with an inexact SQO method
- Allows generic inexactness in QP subproblem solves
- No specific QP solver required
- Global convergence guarantees established
- Numerical experiments suggest inexact algorithm is reliable
- Inexact solutions allowed without degradation of performance

Immediate future work (come to OP14 in San Diego!):

- Comparison with inexact augmented Lagrangian and/or interior-point?
- Benefits of SQO framework? Active-set identification?

Thanks!

"Exact" Algorithms:

- J. V. Burke, F. E. Curtis, and H. Wang, "A Sequential Quadratic Optimization Algorithm with Rapid Infeasibility Detection," in third round of review for *SIAM Journal on Optimization*, originally submitted 2012.
- R. H. Byrd, F. E. Curtis, and J. Nocedal, "Infeasibility Detection and SQP Methods for Nonlinear Optimization," *SIAM Journal on Optimization*, Volume 20, Issue 5, pg. 2281-2299, 2010.

"Inexact" Algorithms:

- F. E. Curtis, T. C. Johnson, D. P. Robinson, and A. Wächter, "An Inexact Sequential Quadratic Optimization Algorithm for Large-Scale Nonlinear Optimization," second round of review for *SIAM Journal on Optimization*, originally submitted 2013.
- F. E. Curtis, J. Huber, O. Schenk, and A. Wächter, "A Note on the Implementation of an Interior-Point Algorithm for Nonlinear Optimization with Inexact Step Computations," *Mathematical Programming, Series B*, Volume 136, Issue 1, pg. 209–227, 2012.
- F. E. Curtis, O. Schenk, and A. Wächter, "An Interior-Point Algorithm for Large-Scale Nonlinear Optimization with Inexact Step Computations," *SIAM Journal on Scientific Computing*, Volume 32, Issue 6, pg. 3447-3475, 2010.
- R. H. Byrd, F. E. Curtis, and J. Nocedal, "An Inexact Newton Method for Nonconvex Equality Constrained Optimization," *Mathematical Programming*, Volume 122, Issue 2, pg. 273-299, 2010.
- F. E. Curtis, J. Nocedal, and A. Wächter, "A Matrix-free Algorithm for Equality Constrained Optimization Problems with Rank-Deficient Jacobians," *SIAM Journal on Optimization*, Volume 20, Issue 3, pg. 1224-1249, 2009.
- R. H. Byrd, F. E. Curtis, and J. Nocedal, "An Inexact SQP Method for Equality Constrained Optimization," *SIAM Journal on Optimization*, Volume 19, Issue 1, pg. 351-369, 2008.